Multivariate Spline Estimation and Inference for Image-On-Scalar Regression

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Abstract: Motivated by recent analyses of data in biomedical imaging studies, we consider a class of image-on-scalar regression models for imaging responses and scalar predictors. We propose using flexible multivariate splines over triangulations to handle the irregular domain of the objects of interest on the images, as well as other characteristics of images. The proposed estimators of the coefficient functions are proved to be root-n consistent and asymptotically normal under some regularity conditions. We also provide a consistent and computationally efficient estimator of the covariance function. Asymptotic pointwise confidence intervals and data-driven simultaneous confidence corridors for the coefficient functions are constructed. Our method can simultaneously estimate and make inferences on the coefficient functions, while incorporating spatial heterogeneity and spatial correlation. A highly efficient and scalable estimation algorithm is developed. Monte Carlo simulation studies are conducted to examine the finite-sample performance of the proposed method, which is then applied to the spatially normalized positron emission tomography data of the Alzheimer's Disease Neuroimaging Initiative.

Key words and phrases: Multivariate splines; Coefficient maps; Confidence corridors; Image Analysis; Triangulation.

1. Introduction

Medical and public health studies collect massive amount of imaging data using methods such as functional magnetic resonance imaging (fMRI), positron emission tomography (PET) imaging, computed tomography (CT), and ultrasonic imaging. Much of these data can be characterized as functional data. Compared with traditional one-dimensional (1D) functional data, these imaging data are complex, high-dimensional, and structured, which poses challenges to traditional statistical methods.

We propose a unifying approach to characterize the varying associations between imaging responses and a set of explanatory variables. Three types of statistical methods are widely used to investigate such associations. The first category includes the univariate approaches and pixel-/voxel-based methods (Worsley et al., 2004; Stein et al., 2010; Hibar et al., 2015), which take each pixel/voxel as a basic analytic unit. Because all pixels/voxels are treated as independent, a major drawback of these methods is that they ignore correlation between

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the pixels/voxels. The second category is the tensor regression. This approach considers an image as a multi-dimensional array (Zhou et al., 2013; Li and Zhang, 2017), which is then changed to a vector to perform the regression. However, doing so naively yields an ultrahigh dimensionality and requires a novel dimension-reduction technique and highly scalable algorithms (Li and Zhang, 2017). The third category is the functional data analysis (FDA) approach, in which an image is viewed as the realization of a function defined on a given domain (Zhu et al., 2012, 2014; Reiss et al., 2017). Using an FDA, we are able to combine information both across and within functions.

We adopt the FDA approach in this study. Functional linear models (FLMs) are widely used to model the regression relationship between a response and some set of predictors from multiple subjects. In the literature (Ramsay and Silverman, 2005; Müller, 2005; Morris, 2015; Wang et al., 2016), FLMs are often categorized based on whether the outcome, the predictor, or both are functional: (i) functional predictor regression (scalar-on-function) (Cardot et al., 1999, 2003; Hall and Horowitz, 2007); (ii) functional response regression (function-on-scalar) (Morris and Carroll, 2006; Reiss et al., 2010; Staicu et al., 2010; Zhu et al., 2014; Zhang and Wang, 2015; Chen et al., 2017); and (iii) function-on-function regression (Ramsay and Dalzell, 1991; Yao et al., 2005; Sentürk and Müller, 2010; Wu and Müller, 2011).

Motivated by the structure of brain imaging data, we propose a novel image-on-scalar regression model with spatially varying coefficients that captures the varying associations between imaging phenotypes and a set of explanatory variables. Figure 1.1 shows a schematic diagram of the proposed modeling approach. Specifically, let Ω be a two-dimensional bounded domain, and let $\mathbf{z} = (z_1, z_2)$ be the location point on Ω . For the *i*th subject, $i = 1, \ldots, n$, let $Y_i(\mathbf{z})$ be the imaging measurement at location $\mathbf{z} \in \Omega$, and let $X_{i\ell}$, for $\ell = 0, 1, \ldots, p$, with $X_{i0} \equiv 1$, be scalar predictors, for example, clinic variables (such as age and sex) and genetic factors. The spatially varying coefficient regression characterizes the associations between imaging measures and covariates, and is given by the following model:

$$Y_i(z) = \widetilde{\mathbf{X}}_i^{\top} \boldsymbol{\beta}^o(z) + \eta_i(z) + \sigma(z) \varepsilon_i(z), \ i = 1, \dots, n, \ z \in \Omega,$$

where $\widetilde{\mathbf{X}}_i = (X_{i0}, X_{i1}, \dots, X_{ip})^{\top}$, $\boldsymbol{\beta}^o = (\beta_0^o, \beta_1^o, \dots, \beta_p^o)^{\top}$ is a vector of some unknown bivariate functions, $\eta_i(\boldsymbol{z})$ characterizes the individual image variations, $\varepsilon_i(\boldsymbol{z})$ represents additional measurement errors, and $\sigma(\boldsymbol{z})$ is a positive deterministic function. In the following, we assume that $\eta_i(\boldsymbol{z})$ and $\varepsilon_i(\boldsymbol{z})$ are mutually independent. Moreover, we assume that $\eta_i(\boldsymbol{z})$, for $i=1,\dots,n$, are independent and identically distributed (i.i.d.) copies of an L_2 stochastic process with mean zero and covariance function $G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') = \text{cov}\{\eta_i(\boldsymbol{z}), \eta_i(\boldsymbol{z}')\}$. Furthermore, $\varepsilon_i(\boldsymbol{z})$, for $i=1,\dots,n$, are i.i.d. copies of a stochastic process with zero mean. and covariance function $G_{\varepsilon}(\boldsymbol{z}, \boldsymbol{z}') = \text{cov}\{\varepsilon_i(\boldsymbol{z}), \varepsilon_i(\boldsymbol{z}')\} = I(\boldsymbol{z}=\boldsymbol{z}')$.

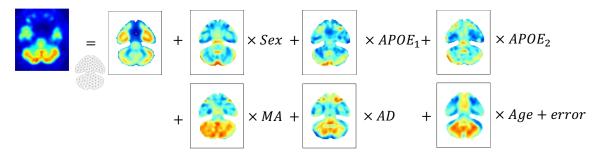


Figure 1.1: A schematic diagram of proposed modeling approach.

For a 1D function-on-scalar regression, Chapter 13 of Ramsay and Silverman (2005) provides a common model-fitting strategy, in which the coefficient functions are expanded using some sets of basis functions, and the basis coefficients are estimated using the ordinary least squares withmethod. However, it is not trivial to extend this to an image-on-scalar regression, particularly with biomedical imaging responses. For biomedical images, the objects (e.g., organs) on the images are usually irregularly shaped (e.g., breast tumors). Another example is that of brain images, as shown in Figure 1.1, especially slices from the bottom and the top of the brain. Even though some images seem to be rectangular, the true signal comes only from the domain of an object, and the image contains only noise outside the boundary of the object. Many smoothing methods, such as, tensor product smoothing (Reiss et al., 2017; Chen et al., 2017), kernel smoothing (Zhu et al., 2014), and wavelet smoothing (Morris and Carroll, 2006), provide poor estimations over difficult regions because they smooth inappropriately across boundary features, referred to as the "leakage" problem in the smoothing literature; see Ramsay (2002) and Sangalli et al. (2013). Next, for technical reasons, imaging data often have different visual qualities. The general characteristics of medical images are determined and limited by the technology for each specific modality. As a result, there is a great interest in developing a flexible method with varying smoothness to adaptively smooth biomedical imaging data.

In this study, we tackle the above challenges using bivariate splines on triangulations (Lai and Wang, 2013) to effectively model the spatially nonstationary relationship and preserve the important features (shape, smoothness) of the imaging data. a triangulation can represent any two-dimensional (2D) geometric domain effectively because any polygon can be decomposed into triangles. We study the asymptotic properties of the bivariate spline estimators of the coefficient functions, and show that our spline estimators are root-n consistent and asymptotically normal. The asymptotic results are used as a guideline to construct pointwise confidence intervals (PCIs) and simultaneous confidence corridors (SCCs; also referred to as "simultaneous confidence bands/regions") for the true coefficient functions. Figure 1.2 shows

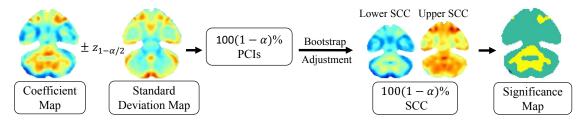


Figure 1.2: A schematic diagram of proposed inferential approach.

the proposed inferential approach. Our method is statistically more efficient than the tensor regression (Li and Zhang, 2017) and the three-stage estimation (Zhu et al., 2014), because it is able to accommodate complex domains of arbitrary shape and adjust the individual smoothing needs of different coefficient functions using multiple smoothing parameters. In addition, our method does not rely on estimating the spatial similarity and adaptive weights repeatedly, as in Zhu et al. (2014); thus, it is much simpler.

The remainder of the paper is structured as follows. Section 2 describes the spline estimators for the coefficient functions, and establishes their asymptotic properties. Section 3 describes the bootstrap method used to construct the SCC and how to estimate the unknown variance functions involved in the SCC. Section 4 presents the implementation of the proposed estimation and inference. Section 5 reports our findings from two simulation studies. In Section 6, we illustrate the proposed method using PET data provided by the Alzheimer's Disease Neuroimaging Initiative (ADNI). Section 7 concludes the paper. All technical proofs of the theoretical results and additional numerical results are deferred to the Appendices A and B.

2. Models and Estimation Method

2.1. Image-on-scalar regression model

Let $z_j \in \Omega$ be the center point of the jth pixel in the domain Ω , and let Y_{ij} be the imaging response of subject i at location j. the actual data set consists of $\{(Y_{ij}, \widetilde{\mathbf{X}}_i, \mathbf{z}_j), i = 1, \dots, n, j = 1, \dots, N\}$, which can be modeled as follows:

$$Y_{ij} = \sum_{\ell=0}^{p} X_{i\ell} \beta_{\ell}^{o}(\boldsymbol{z}_{j}) + \eta_{i}(\boldsymbol{z}_{j}) + \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij}.$$

$$(2.1)$$

Denote the eigenvalues and eigenfunctions of the covariance operator $G_{\eta}(z, z')$ as $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\psi_k(z)\}_{k=1}^{\infty}$, respectively, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\sum_{k=1}^{\infty} \lambda_k < \infty$, and $\{\psi_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$. It follows from spectral theory that $G_{\eta}(z, z') = \sum_{k=1}^{\infty} \lambda_k \psi_k(z) \psi_k(z')$. The *i*th trajectory $\{\eta_i(z), z \in \Omega\}$ allows the Karhunen–Loéve L^2 representation (Li and Hsing, 2010; Sang and Huang, 2012): $\eta_i(z) = \sum_{k=1}^{\infty} \lambda_k^{1/2} \xi_{ik} \psi_k(z)$, $\lambda_k^{1/2} \xi_{ik} = \int_{z \in \Omega} \eta_i(z) \psi_k(z) dz$, where the random coefficients ξ_{ik} are uncorrelated random variables with mean zero and

 $E(\xi_{ik}\xi_{ik'}) = I(k = k')$, referred to as the kth functional principal component score (FPCA) of the ith subject. Thus, the response measurements in (2.1) can be represented as follows:

$$Y_{ij} = \sum_{\ell=0}^{p} \beta_{\ell}^{o}(\boldsymbol{z}_{j}) X_{i\ell} + \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) + \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij}.$$
 (2.2)

2.2. Spline approximation over triangulations and penalized regression

Note that the objects of interest on many biomedical images are often distributed over an irregular domain Ω . Triangulation is an effective strategy to handle such data. For example, the spatial smoothing problem over difficult regions in Ramsay (2002) and Sangalli et al. (2013) was solved using the finite element method (FEM) on triangulations, which was developed primarily to solve partial differential equations. Here, we approximate each coefficient function in (2.2) using bivariate splines over triangulations (Lai and Schumaker, 2007). The idea is to approximate each function $\beta_{\ell}(\cdot)$ using Bernstein basis polynomials that are piecewise polynomial functions over a 2D triangulated domain. Compared with the FEM, the proposed approach is appealing in the sense that its spline functions are more flexible and it uses various smoothness settings to better approximate the coefficient functions. In this section, we briefly introduce the triangulation technique and describe the bivariate penalized spline smoothing (BPST) method used to approximate the spatial data.

Triangulation is an effective tool to deal with data distributed over difficult regions with complex boundaries and/or interior holes. In the following, we use T to denote a triangle that is a convex hull of three points not located on one line. A collection $\triangle = \{T_1, \dots, T_H\}$ of H triangles is called a triangulation of $\Omega = \bigcup_{h=1}^{H} T_h$, provided that any nonempty intersection between a pair of triangles in \triangle is either a shared vertex or a shared edge. Given a triangle $T \in \Delta$, let |T| be its longest edge length and ϱ_T be the radius of the largest disk inscribed in T. Define the shape parameter of T as the ratio $\pi_T = |T|/\varrho_T$. When π_T is small, the triangles are relatively uniform in the sense that all angles of the triangles in \triangle are relatively the same. Denote the size of \triangle by $|\triangle| = \max\{|T|, T \in \triangle\}$, that is, the length of the longest edge of \triangle . For an integer $r \geq 0$, let $\mathcal{C}^r(\Omega)$ be the collection of all rth continuously differentiable functions over Ω . Given \triangle , let $\mathcal{S}_d^r(\triangle) = \{s \in \mathcal{C}^r(\Omega) : s|_T \in \mathbb{P}_d(T), T \in \triangle\}$ be a spline space of degree dand smoothness r over \triangle , where $s|_T$ is the polynomial piece of spline s restricted on triangle T, and \mathbb{P}_d is the space of all polynomials of degree less than or equal to d. Note that the major difference between the FEM and the BPST is the flexibility of the smoothness, r, and the degree of the polynomials, d. Specifically, the FEM in Sangalli et al. (2013) requires that r=0 and d=1 or 2, whereas the BPST allows smoothness $r\geq 0$ and various degrees of polynomials.

We use Bernstein basis polynomials to represent the bivariate splines. For any ℓ

 $0, 1, \ldots, p$, denote by Δ_{ℓ} the triangulation of the ℓ th component. Define

$$\mathcal{G}^{(p+1)} \equiv \mathcal{G}^{(p+1)}(\triangle_0 \times \cdots \times \triangle_p) = \left\{ \boldsymbol{g} = (g_0, \dots, g_p)^\top, g_\ell \in \mathcal{S}^r_d(\triangle_\ell), \ell = 0, \dots, p \right\}$$

and let $\{B_{\ell m}\}_{m\in\mathcal{M}_{\ell}}$ be the set of degree-d bivariate Bernstein basis polynomials for $\mathcal{S}_{d}^{r}(\triangle_{\ell})$, where \mathcal{M}_{ℓ} is an index set of Bernstein basis polynomials. Denote by \mathbf{B}_{ℓ} the evaluation matrix of the Bernstein basis polynomials for the ℓ th component, and let the jth row of \mathbf{B}_{ℓ} is given by $\mathbf{B}_{\ell}^{\top}(\mathbf{z}_{j}) = \{B_{\ell m}(\mathbf{z}_{j}), m \in \mathcal{M}_{\ell}\}$. We approximate each $\beta_{\ell}(\cdot)$ using $\beta_{\ell}(\mathbf{z}_{j}) \approx \mathbf{B}_{\ell}^{\top}(\mathbf{z}_{j})\gamma_{\ell}$, for $\ell = 0, 1, \ldots, p$, where $\gamma_{\ell}^{\top} = (\gamma_{\ell m}, m \in \mathcal{M}_{\ell})$ is the spline coefficient vector.

Penalized spline smoothing has gained in popularity over the last two decades; see Hall and Opsomer (2005); Claeskens et al. (2009); Schwarz and Krivobokova (2016). To define the penalized spline method, for any direction z_q , q = 1, 2, let $\nabla_{z_q}^v s(z)$ denote the vth-order derivative in the direction z_q at the point z. We consider the following penalized least squares problem:

$$\min_{(\beta_0,\dots,\beta_p)^\top \in \mathcal{G}^{(p+1)}} \sum_{i=1}^n \sum_{j=1}^N \left\{ Y_{ij} - \sum_{\ell=0}^p X_{i\ell} \beta_\ell(\boldsymbol{z}_j) \right\}^2 + \sum_{\ell=0}^p \rho_{n,\ell} \mathcal{E}(\beta_\ell),$$

where $\mathcal{E}(s) = \sum_{T \in \triangle} \int_T \sum_{i+j=2} {2 \choose i} (\nabla_{z_1}^i \nabla_{z_2}^j s)^2 dz_1 dz_2$ is the roughness penalty, and $\rho_{n,\ell}$ is the penalty parameter for the ℓ th function.

To satisfy the smoothness condition of the splines, we need to impose some linear constraints on the spline coefficients γ_{ℓ} : $\mathbf{H}_{\ell}\gamma_{\ell} = \mathbf{0}$, for $\ell = 0, 1, \dots, p$. Thus, we have to minimize the following constrained least squares:

$$\sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ Y_{ij} - \sum_{\ell=0}^{p} X_{i\ell} \mathbf{B}_{\ell}^{\top}(\boldsymbol{z}_{j}) \boldsymbol{\gamma}_{\ell} \right\}^{2} + \sum_{\ell=0}^{p} \rho_{n,\ell} \boldsymbol{\gamma}_{\ell}^{\top} \mathbf{P}_{\ell} \boldsymbol{\gamma}_{\ell}, \text{ subject to } \mathbf{H}_{\ell} \boldsymbol{\gamma}_{\ell} = 0,$$

where \mathbf{P}_{ℓ} is the block diagonal penalty matrix satisfying $\boldsymbol{\gamma}_{\ell}^{\top} \mathbf{P}_{\ell} \boldsymbol{\gamma}_{\ell} = \mathcal{E}(\mathbf{B}_{\ell}^{\top} \boldsymbol{\gamma}_{\ell})$.

We first remove the constraint using a QR decomposition of the transpose of the constraint matrix \mathbf{H}_{ℓ} . Applying a QR decomposition on \mathbf{H}_{ℓ}^{\top} , we have $\mathbf{H}_{\ell}^{\top} = \mathbf{Q}_{\ell}\mathbf{R}_{\ell} = (\mathbf{Q}_{\ell,1}\ \mathbf{Q}_{\ell,2})\binom{\mathbf{R}_{\ell,1}}{\mathbf{R}_{\ell,2}}$, where \mathbf{Q}_{ℓ} is an orthogonal matrix and \mathbf{R}_{ℓ} is an upper triangular matrix. the submatrix $\mathbf{Q}_{\ell,1}$ represents the first r columns of \mathbf{Q}_{ℓ} , where r is the rank of matrix \mathbf{H}_{ℓ} , and $\mathbf{R}_{\ell,2}$ is a matrix of zeros. We reparametrize this using $\gamma_{\ell} = \mathbf{Q}_{\ell,2}\boldsymbol{\theta}_{\ell}$, for some $\boldsymbol{\theta}_{\ell}$. Then, it is guaranteed that $\mathbf{H}_{\ell}\gamma_{\ell} = \mathbf{0}$. Thus, the minimization problem is converted to the following conventional penalized regression problem, without restrictions:

$$\sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ Y_{ij} - \sum_{\ell=0}^{p} X_{i\ell} \mathbf{B}_{\ell}^{\mathsf{T}}(\boldsymbol{z}_{j}) \mathbf{Q}_{\ell,2} \boldsymbol{\theta}_{\ell} \right\}^{2} + \sum_{\ell=0}^{p} \rho_{n,\ell} \boldsymbol{\theta}_{\ell}^{\mathsf{T}} \mathbf{D}_{\ell} \boldsymbol{\theta}_{\ell}, \tag{2.3}$$

where $\mathbf{D}_{\ell} = \mathbf{Q}_{\ell,2}^{\mathsf{T}} \mathbf{P}_{\ell} \mathbf{Q}_{\ell,2}$.

Let $\widetilde{\mathbf{Y}}_i = (Y_{i1}, Y_{i2}, \dots, Y_{iN})^{\top}$, $\mathbf{B}_{\ell}(\mathbf{z}) = \{B_{\ell m}(\mathbf{z}), m \in \mathcal{M}_{\ell}\}^{\top}$, $\mathbb{Y} = (\widetilde{\mathbf{Y}}_1^{\top}, \dots, \widetilde{\mathbf{Y}}_n^{\top})^{\top}$, and $\mathbb{U} = (\mathbf{U}_{11}, \mathbf{U}_{12}, \dots, \mathbf{U}_{nN})^{\top}$, where

$$\mathbf{U}_{ij} = \{X_{i0}\mathbf{B}_0(\boldsymbol{z}_j)^{\top}\mathbf{Q}_{0,2}, X_{i1}\mathbf{B}_1(\boldsymbol{z}_j)^{\top}\mathbf{Q}_{1,2}, \cdots, X_{ip}\mathbf{B}_p(\boldsymbol{z}_j)^{\top}\mathbf{Q}_{p,2}\}^{\top}.$$
 (2.4)

Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_0^{\top}, \boldsymbol{\theta}_1^{\top}, \dots, \boldsymbol{\theta}_p^{\top})^{\top}$ and $\mathbb{D}(\rho_{n,0}, \dots, \rho_{n,p}) = \operatorname{diag}\{\rho_{n,0}\mathbf{D}_0, \dots, \rho_{n,p}\mathbf{D}_p\}$. Minimizing (2.3) is then equivalent to minimizing $\|\mathbb{Y} - \mathbb{U}\boldsymbol{\theta}\|^2 + \boldsymbol{\theta}^{\top}\mathbb{D}(\rho_{n,0},\ldots,\rho_{n,p})\boldsymbol{\theta}$. Hence,

$$\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}_0^\top, \widehat{\boldsymbol{\theta}}_1^\top, \dots, \widehat{\boldsymbol{\theta}}_p^\top)^\top = \{\mathbb{U}^\top \mathbb{U} + \mathbb{D}(\rho_{n,0}, \dots, \rho_{n,p})\}^{-1} \mathbb{U}^\top \mathbb{Y}.$$

Thus, the estimators of γ_{ℓ} and $\beta_{\ell}(\cdot)$ are

$$\widehat{\gamma}_{\ell} = \mathbf{Q}_{\ell,2}\widehat{\boldsymbol{\theta}}_{\ell}, \quad \widehat{\beta}_{\ell}(\boldsymbol{z}) = \mathbf{B}_{\ell}(\boldsymbol{z})^{\top}\widehat{\boldsymbol{\gamma}}_{\ell}.$$
 (2.5)

2.3. Asymptotic properties of the BPST estimators

This section examines the asymptotics of the proposed estimators. Given random variables U_n for $n \ge 1$, we write $U_n = O_P(b_n)$ if $\lim_{c \to \infty} \limsup_n P(|U_n| \ge cb_n) = 0$. Similarly, we write $U_n = o_P(b_n)$ if $\lim_n P(|U_n| \ge cb_n) = 0$, for any constant c > 0. Next, to facilitate discussion, we introduce some notation of norms. For any function g over the closure of domain Ω , denote $||g||_{L^2(\Omega)}^2 = \int_{\Omega} g^2(z) dz$ as the regular L_2 norm of g, and $||g||_{\infty,\Omega} = \sup_{z \in \Omega} |g(z)|$ as the supremum norm of g. Further denote $\|\mathbf{g}\|_{v,\infty,\Omega} = \max_{0 \le \ell \le p} |g_{\ell}|_{v,\infty,\Omega}$, where $|g|_{v,\infty,\Omega} = \max_{0 \le \ell \le p} |g_{\ell}|_{v,\infty,\Omega}$ $\max_{i+j=v} \|\nabla_{z_1}^i \nabla_{z_2}^j g\|_{\infty,\Omega}$ is the maximum norm of all vth-order derivatives of g over Ω . Let $\mathcal{W}^{d,\infty}(\Omega)=\{g:|g|_{k,\infty,\Omega}<\infty,0\leq k\leq d\}$ be the standard Sobolev space. Next, we introduce some technical conditions.

- (A1) For any $\ell = 0, \dots, p$, $\beta_{\ell}^{o}(\cdot) \in \mathcal{W}^{d+1,\infty}(\Omega)$, for an integer $d \geq 1$. (A2) For any $i = 1, \dots, n, j = 1, \dots, N$, ε_{ij} 's are independent with mean zero and variance one, and for any $k \geq 1$, ξ_{ik} are uncorrelated random variables with mean zero and variance
- (A3) For any $\ell = 0, 1, ..., p$, there exists a positive constant C_{ℓ} , such that $E|X_{\ell}|^{8} \leq C_{\ell}$. The eigenvalues of $\Sigma_X = E(XX^{\top})$ are bounded away from zero and infinity. (A4) The function $\sigma(z) \in \mathcal{C}^{(1)}(\Omega)$, with $0 < c_{\sigma} \le \sigma(z) \le C_{\sigma} \le \infty$, for any $z \in \Omega$; for any k,
- $\psi_k(z) \in \mathcal{C}^{(1)}(\Omega)$ and $0 < c_G \le G_\eta(z, z) \le C_G \le \infty$, for any $z \in \Omega$. (A5) Let $|\underline{\triangle}| = \min_{0 \le \ell \le p} |\Delta_\ell|$ and $|\overline{\triangle}| = \max_{0 \le \ell \le p} |\Delta_\ell|$. the triangulations Δ_ℓ satisfy that $\limsup_n (|\bar{\Delta}|/|\Delta|) < \infty$. The triangulations are π -quasi-uniform; that is, there exists a
- positive constant π , such that $\max_{0 \leq \ell \leq p} \{ (\min_{T \in \triangle_{\ell}} \varrho_T)^{-1} |\Delta_{\ell}| \} \leq \pi$. (A6) As $N \to \infty$, $n \to \infty$, for some $0 < \kappa < 1$, $N^{-1} n^{1/(d+1) + \kappa} \to 0$, $n^{1/2} |\overline{\triangle}|^{d+1} \to 0$, $N^{1/2}|\underline{\triangle}| \to \infty$, and the smoothing parameters satisfy that $n^{-1/2}N^{-1}|\underline{\triangle}|^{-3}\rho_n \to 0$, where $\rho_n = \max_{0 \le \ell \le p} \rho_{n,\ell}.$

The above assumptions are mild conditions that are satisfied in many practical situations. Assumption (A1) describes the usual requirement on the coefficient functions described in the literature on nonparametric estimation. Assumption (A1) can be relaxed to Assumption (A1') in Section 2.4, which only requires $\beta_\ell^o(\cdot) \in \mathcal{C}^{(0)}(\Omega)$ when dealing with imaging data with sharp edges; see Section 2.4. Assumptions (A1) and (A2) are similar to Assumptions (A1) and (A2) in Gu et al. (2014) and Assumptions (A1)-(A3) in Huang et al. (2004). Assumption (A3) is analogous to Assumption (A5) in Gu et al. (2014), ensuring that $X_{i\ell}$ is not multicollinear. Assumption (A5) requires that Δ_ℓ be of similar size, and suggests the use of more uniform triangulations with smaller shape parameters. Assumption (A6) implies that the number of pixels for each image N diverges to infinity and the sample size n grows as $N \to \infty$, a well-developed asymptotic scenario for dense functional data (Li and Hsing, 2010). Assumption (A6) also describes the requirement of the growth rate of the dimension of the spline spaces relative to the sample size and the image resolution. This assumption is easily satisfied because images measured using current technology are usually of sufficiently high resolution.

The following theorem provides the L_2 convergence rate of $\widehat{\beta}_{\ell}(\cdot)$, for $\ell = 0, 1, \dots, p$. a detailed proof is given in Appendix A.

Theorem 1. Suppose Assumptions (A1)-(A5) hold and $N^{1/2}|\underline{\triangle}| \to \infty$ as $N \to \infty$. Then, for any $\ell = 0, 1, \ldots, p$, the BPST estimator $\widehat{\beta}_{\ell}(\cdot)$ is consistent and satisfies $\|\widehat{\beta}_{\ell} - \beta^{o}_{\ell}\|_{L^{2}(\Omega)} = O_{P}\left\{\frac{\rho_{n}}{nN|\Delta|^{3}}\|\boldsymbol{\beta}^{o}\|_{2,\infty} + \left(1 + \frac{\rho_{n}}{nN|\Delta|^{5}}\right)|\overline{\Delta}|^{d+1}\|\boldsymbol{\beta}^{o}\|_{d+1,\infty} + n^{-1/2}\right\}.$

Theorem 2 states the asymptotic normality of $\widehat{\beta}_{\ell}$ at any given point $z \in \Omega$, for $\ell = 0, 1, \ldots, p$. See Appendix A for a detailed proof. Denote

$$\mathbf{\Xi}_{n}(\boldsymbol{z}) = \widetilde{\mathbb{B}}(\boldsymbol{z})^{\top} E \left\{ \mathbf{\Gamma}_{n,\rho}^{-1} \frac{1}{n^{2} N^{2}} \sum_{i=1}^{n} \sum_{j,j'=1}^{N} \mathbf{U}_{ij} \mathbf{U}_{ij'}^{\top} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) \mathbf{\Gamma}_{n,\rho}^{-1} \right\} \widetilde{\mathbb{B}}(\boldsymbol{z}),$$
(2.6)

where \mathbf{U}_{ij} and $\mathbf{\Gamma}_{n,\rho}$ are given in (2.4) and (A.17), respectively, in Appendix 1, $\widetilde{\mathbf{B}}_{\ell}(\boldsymbol{z}) = \mathbf{Q}_{2,\ell}^{\top} \mathbf{B}_{\ell}(\boldsymbol{z})$ for $\ell = 0, \dots, p$, and $\widetilde{\mathbb{B}}(\boldsymbol{z}) = \operatorname{diag}\{\widetilde{\mathbf{B}}_{0}(\boldsymbol{z}), \dots, \widetilde{\mathbf{B}}_{p}(\boldsymbol{z})\}.$

Theorem 2. Suppose Assumptions (A1)-(A6) hold. If for any $\ell = 0, 1, ..., p$, $|X_{i\ell}| \leq C_{\ell} < \infty$, then $\Xi_n^{-1/2}(z)\{\widehat{\beta}(z) - \beta^o(z)\} \stackrel{\mathcal{L}}{\longrightarrow} N\left(\mathbf{0}, \mathbf{I}_{(p+1)\times(p+1)}\right)$ as $N \to \infty$ and $n \to \infty$, where $\Xi_n(z)$ is given in (2.6). Furthermore, there exist positive constants $c_V < C_V < +\infty$, such that $c_V n^{-1} \left(1 + \frac{\rho_n}{nN|\triangle|^4}\right)^{-2} \leq \operatorname{Var}\{\widehat{\beta}_{\ell}(z)\} \leq C_V n^{-1}$, for any $\ell = 0, 1, ..., p$.

2.4. Piecewise constant spline over triangulation smoothing

Many imaging data can be regarded as a noisy version of a piecewise-smooth function of $z \in \Omega$ with sharp edges, which often reflect the functional or structural changes. The

penalized bivariate spline smoothing method introduced, in Section 2.2, assumes some degree of smoothness over the entire image. To relax this assumption while preserving the features of sharp edges, we make the following less stringent assumption on the smoothness of the coefficient functions:

(A1') For any $\ell = 0, \dots, p$, the bivariate function $\beta_{\ell}^{o}(\cdot) \in \mathcal{C}^{(0)}(\Omega)$.

For the estimation, we consider the piecewise constant spline over triangulation (PCST) method. For any $\ell = 1, ..., p$, denote by $\mathcal{PC}(\triangle_{\ell})$ the space of piecewise constant functions over each T_m , for $m \in \mathcal{M}_{\ell}$. The bivariate spline basis functions of $\mathcal{PC}(\triangle_{\ell})$ are denoted as $\{B_{\ell m}(z)\}_{m \in \mathcal{M}_{\ell}}$, which are simply indicator functions over triangle T_m , $B_{\ell m}(z) = I(z \in T_m)$, $m \in \mathcal{M}_{\ell}$. Assumption (A1') controls the bias of the piecewise constant spline estimator for β_{ℓ}^{o} and leads to the estimation consistency.

When using the constant bivariate spline basis functions, we have $\mathcal{E}(s) = 0$ for all $s \in \mathcal{PC}(\Delta)$, and for any $\mathbf{z} \in \Omega$, $\mathbf{B}_{\ell}(\mathbf{z})\mathbf{B}_{\ell}(\mathbf{z})^{\top} = \operatorname{diag}\{B_{\ell m}^{2}(\mathbf{z}), m \in \mathcal{M}_{\ell}\}$. Then, we can write $\widehat{\boldsymbol{\gamma}}_{m} = (\widehat{\gamma}_{0m}, \widehat{\gamma}_{1m}, \dots, \widehat{\gamma}_{pm})^{\top} = \widehat{\mathbf{V}}_{m}^{-1} \left\{ (nN)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{\ell m}(\mathbf{z}_{j}) X_{i\ell} Y_{ij} \right\}_{\ell=0}^{p}$, where

$$\widehat{\mathbf{V}}_{m} = \frac{1}{nN} \sum_{j=1}^{N} B_{\ell m}^{2}(\mathbf{z}_{j}) \sum_{i=1}^{n} \widetilde{\mathbf{X}}_{i} \widetilde{\mathbf{X}}_{i}^{\top} = \left\{ \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{\ell m}^{2}(\mathbf{z}_{j}) X_{i\ell} X_{i\ell'} \right\}_{\ell,\ell'=0}^{p}.$$
 (2.7)

By simple linear algebra, for any $\ell = 0, \dots, p$, the PCST estimator is given by

$$\widehat{\beta}_{\ell}^{c}(z) = \sum_{m \in \mathcal{M}_{\ell}} \widehat{\gamma}_{\ell m} B_{\ell m}(z). \tag{2.8}$$

For any $z \in \Omega$, define the index of the triangle containing z as m(z); that is, m(z) = m if $z \in T_m$. Then, $\widehat{\beta}_{\ell}(z) = \widehat{\gamma}_{\ell m(z)}$ and $\widehat{\beta}^{c}(z) = (\widehat{\beta}_{0}^{c}(z), \dots, \widehat{\beta}_{p}^{c}(z))^{\top} = (\widehat{\gamma}_{0m(z)}, \dots, \widehat{\gamma}_{pm(z)})^{\top} = \widehat{\gamma}_{m(z)}$. For any $z \in \Omega$, denote

$$\Sigma_n(\mathbf{z}) = n^{-1} \Sigma_X^{-1} G_\eta(\mathbf{z}, \mathbf{z}). \tag{2.9}$$

Theorem 3 shows the asymptotic normality of the piecewise constant estimators $\beta(z)$. See the Appendix A for detailed proofs. To obtain the asymptotic variance-covariance function, we also need the following assumption:

(C1) The variables ξ_{ik} and ε_{ij} are independent and satisfy $E |\xi_{ik}|^{4+\delta_1} < +\infty$ for some $\delta_1 > 0$, and $E |\varepsilon_{ij}|^{4+\delta_2} < \infty$ for some $\delta_2 > 0$.

Theorem 3. Under Assumptions (A1'), (A2)-(A5), and (C1), as $N \to \infty$ and $n \to \infty$, if for some $0 < \kappa < 1$, $N^{-1}n^{1+\kappa} \to 0$, $N^{-1/2} \ll |\underline{\triangle}| \le |\overline{\triangle}| \ll n^{1/4}N^{-1/2}$, and $\|\sum_{k=1}^{\infty} \lambda_k^{1/2} \psi_k\|_{\infty} < 1$

 ∞ , then for any $\mathbf{z} \in \Omega$, $\mathbf{\Sigma}_n^{-1/2}(\mathbf{z})\{\widehat{\boldsymbol{\beta}}^c(\mathbf{z}) - \boldsymbol{\beta}^o(\mathbf{z})\} \xrightarrow{\mathcal{L}} N\left(\mathbf{0}, \mathbf{I}_{(p+1)\times(p+1)}\right)$, where $\mathbf{\Sigma}_n(\mathbf{z})$ is (2.9); $\operatorname{pr}\left\{(\sigma_{n,\ell\ell}^c)^{-1}(\mathbf{z}) \left| \widehat{\boldsymbol{\beta}}_{\ell}(\mathbf{z}) - \boldsymbol{\beta}_{\ell}^o(\mathbf{z}) \right| \leq Z_{1-\alpha/2}\right\} \to 1-\alpha$, for any $\alpha \in (0,1)$, as $N \to \infty$, $n \to \infty$, where $\sigma_{n,\ell\ell}^c(\mathbf{z})$ is the square root of the (ℓ,ℓ) th entry of the matrix $\mathbf{\Sigma}_n(\mathbf{z})$, and $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution.

3. Variance Function Estimation and Simultaneous Confidence Corridors

3.1. Estimation of the variance function

Define the estimated residual $\widehat{R}_{ij} = Y_{ij} - \sum_{\ell=0}^p X_{i\ell} \widehat{\beta}_{\ell}(\boldsymbol{z}_j)$ or $Y_{ij} - \sum_{\ell=0}^p X_{i\ell} \widehat{\beta}_{\ell}^c(\boldsymbol{z}_j)$, for any $i = 1, \ldots, n, \ j = 1, \ldots, N$. We apply the bivariate spline smoothing method to $\{(\widehat{R}_{ij}, \boldsymbol{z}_j)\}_{j=1}^N$. Specifically, we define

$$\widehat{\eta}_i(\boldsymbol{z}) = \operatorname*{arg\,min}_{g_i \in \mathcal{S}_d^r(\Delta_\eta)} \sum_{j=1}^N \left\{ \widehat{R}_{ij} - g_i(\boldsymbol{z}_j) \right\}^2, \ i = 1, \dots, n,$$
(3.1)

as the spline estimator of $\eta_i(z)$, where the triangulation \triangle_{η} may differ from that introduced in Section 2 when estimating $\beta_{\ell}^{o}(z)$. Next, let $\hat{\epsilon}_{ij} = \hat{R}_{ij} - \hat{\eta}_{i}(z_{j})$. Define the estimators of $G_{\eta}(z, z')$ and $\sigma^{2}(z_{j})$ as

$$\widehat{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}') = n^{-1} \sum_{i=1}^{n} \widehat{\eta}_{i}(\boldsymbol{z}) \widehat{\eta}_{i}(\boldsymbol{z}') \text{ and } \widehat{\sigma}^{2}(\boldsymbol{z}_{j}) = n^{-1} \sum_{i=1}^{n} \widehat{\epsilon}_{ij} \widehat{\epsilon}_{ij},$$
(3.2)

respectively. In general, for spline estimators $(d \ge 0)$, denote $\widehat{\Xi}_n(z) = \left\{\widehat{\sigma}_{n,\ell\ell'}^2(z)\right\}_{\ell,\ell'=0}^p$, where

$$\widehat{\boldsymbol{\Xi}}_{n}(\boldsymbol{z}) = \frac{1}{n^{2}N^{2}} \widetilde{\mathbb{B}}(\boldsymbol{z})^{\top} \sum_{i=1}^{n} \left\{ \sum_{j,j'=1}^{N} \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{U}_{ij} \mathbf{U}_{ij'}^{\top} \widehat{G}_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) \boldsymbol{\Gamma}_{n,\rho}^{-1} + \sum_{j=1}^{N} \mathbf{U}_{ij} \mathbf{U}_{ij}^{\top} \widehat{\sigma}^{2}(\boldsymbol{z}_{j}) \right\} \widetilde{\mathbb{B}}(\boldsymbol{z}). \quad (3.3)$$

Note that the estimation can be much simplified if PCST smoothing is applied. In this case, the variance-covariance matrix $\Sigma_n(z)$ can be simply estimated using

$$\widehat{\boldsymbol{\Sigma}}_n(\boldsymbol{z}) = \left\{ (\widehat{\boldsymbol{\sigma}}_{n,\ell\ell'}^{\text{c}})^2(\boldsymbol{z}) \right\}_{\ell,\ell'=0}^p = \frac{1}{n} \left(n^{-1} \sum_{i=1}^n \widetilde{\mathbf{X}}_i \widetilde{\mathbf{X}}_i^\top \right)^{-1} \left\{ \widehat{\boldsymbol{G}}_{\eta}(\boldsymbol{z},\boldsymbol{z}) + \frac{\widehat{\boldsymbol{\sigma}}^2(\boldsymbol{z})}{NA_{m(\boldsymbol{z})}} \right\},$$

where $A_{m(z)}$ is the area of triangle $T_{m(z)}$ divided by the area of the domain. The following conditions (C2)–(C3) are required for the bivariate spline approximation in the covariance estimation and to establish the estimation consistency. The proofs of the results in this section are provided in the Appendix A.

(C2) For any $k \geq 1$, $\psi_k(z) \in \mathcal{W}^{s+1,\infty}$ for an integer $s \geq 0$, and for a sequence $\{K_n\}_{n=1}^{\infty}$ of increasing positive integers with $\lim_n K_n \to \infty$, $|\Delta_{\eta}|^{s+1} \sum_{k=1}^{K_n} \lambda_k^{1/2} \|\psi_k\|_{s+1,\infty} \to 0$ as $N \to \infty$, $n \to \infty$.

(C3) As $N \to \infty$, $n \to \infty$, for some $0 < \kappa < 1$, $N^{-1}n^{1/(d+1)+\kappa} \to 0$, $N|\triangle_{\eta}|^2 \to \infty$, and $n|\triangle_{\eta}|^2/(\log n)^{1/2} \to \infty$.

Assumption (C2) concerns the bounded smoothness of the principal components that bound the bias terms in the spline covariance estimator.

Theorem 4. Under Assumptions (A1)-(A6) and (C1)-(C3), $\widehat{G}_{\eta}(z,z')$ uniformly converges to $G_{\eta}(z,z')$ in probability; that is, $\sup_{(z,z')\in\Omega^2}|\widehat{G}_{\eta}(z,z')-G_{\eta}(z,z')|=o_P(1)$.

Corollary 1. Under Assumptions (A1)-(A6), (C1)-(C3), the estimator of $\widehat{\Sigma}_n(z)$ uniformly converges to to $\Sigma_n(z)$ in probability; that is, $\sup_{z\in\Omega}|\widehat{\Sigma}_n(z)-\Sigma_n(z)|=o_P(1)$.

Denote

$$\widehat{\sigma}_{n,\ell\ell}^{c}(\boldsymbol{z}) = n^{-1/2} \left[\mathbf{e}_{\ell}^{\top} \left(n^{-1} \sum_{i=1}^{n} \widetilde{\mathbf{X}}_{i} \widetilde{\mathbf{X}}_{i}^{\top} \right)^{-1} \mathbf{e}_{\ell} \left\{ \widehat{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}) + \frac{\widehat{\sigma}^{2}(\boldsymbol{z})}{N A_{m(\boldsymbol{z})}} \right\} \right]^{1/2}.$$
(3.4)

From Corollary 1, $\hat{\sigma}_{n,\ell\ell}^{c}(z)$ is a consistent estimator of $\sigma_{n,\ell\ell}^{c}(z)$ in (2.9).

3.2. Bootstrap simultaneous confidence corridors (SCCs)

From Theorems 2 and 3 and Slutzky's Theorem, we have the following asymptotic PCIs.

Corollary 2. (a) For the BPST estimators, under Assumptions (A1)-(A6), for any $\ell = 0, \ldots, p$, $\alpha \in (0,1)$, as $N \to \infty$, $n \to \infty$, an asymptotic $100(1-\alpha)\%$ PCI for $\beta_{\ell}^{o}(\mathbf{z})$, is $\widehat{\beta}_{\ell}(\mathbf{z}) \pm \sigma_{n,\ell\ell}(\mathbf{z}) Z_{1-\alpha/2}$, for any $\mathbf{z} \in \Omega$, where $\sigma_{n,\ell\ell}^2(\mathbf{z})$ is the (ℓ,ℓ) th entry of the matrix $\mathbf{\Xi}_n^{-1/2}(\mathbf{z})$, and $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution.

(b) For the PCST estimators, under Assumptions (A1') and (A2)-(A6), if for some $0 < \kappa < 1$, $N^{-1}n^{1+\kappa} \to 0$, an asymptotic $100(1-\alpha)\%$ PCI for $\beta_\ell^o(z)$ is $\widehat{\beta}_\ell^c(z) \pm \sigma_{n,\ell\ell}^c(z) Z_{1-\alpha/2}$, for any $z \in \Omega$, where $\sigma_{n,\ell\ell}^c(z)$ is the standard deviation function of $\widehat{\beta}_\ell^c(z)$ in Theorem 3.

Next, we introduce a simple bootstrap approach to extend the PCIs to the SCCs. Our approach is based on the nonparametric bootstrap method used in Hall and Horowitz (2013). We triangulate the domain Ω using quasi-uniform triangles, obtaining a set of approximate $100(1-\alpha)\%$ PCIs. In the following, α_0 denotes the nominal confidence level of the desired SCCs. We recalibrate the PCIs using the following bootstrap method.

Step 1. Based on $\left\{ (\widetilde{\mathbf{X}}_i, Y_{ij}) \right\}_{j=1,i=1}^{N,n}$, obtain the coefficient functions $\beta_\ell^o(\boldsymbol{z})$ using the BPST estimators $\widehat{\beta}_\ell^c(\boldsymbol{z})$ in (2.5) or the PCST estimators $\widehat{\beta}_\ell^c(\boldsymbol{z})$ in (2.8), for $\ell = 0, \dots, p$. Let $\widehat{\mu}(\boldsymbol{z}) = \sum_{\ell=0}^p X_{i\ell} \widehat{\beta}_\ell(\boldsymbol{z})$ or $\sum_{\ell=0}^p X_{i\ell} \widehat{\beta}_\ell^c(\boldsymbol{z})$.

- Step 2. Obtain $\widehat{\eta}_i(z)$ and $\widehat{\varepsilon}_{ij}$ presented in (3.1)–(3.2), and estimate $G_{\eta}(z,z)$, $\sigma^2(z)$, and $\sigma^2_{n,\ell\ell}(z)$ using $\widehat{G}_{\eta}(z,z)$ and $\widehat{\sigma}^2(z)$ in (3.2) and $\widehat{\sigma}^2_{n,\ell\ell}(z)$ in (3.3) or (3.4), respectively.
- Step 3. Obtain an adjusted nominal confidence level $\hat{\alpha}_{\ell}(\alpha_0)$.
 - (i) Generate an independent random sample $\delta_i^{(b)}$ and $\delta_{ij}^{(b)}$ from $\{-1,1\}$ with probability 0.5 each, and define $Y_{ij}^{*(b)} = \widehat{\mu}(\boldsymbol{z}_j) + \delta_i^{(b)} \widehat{\eta}_i(\boldsymbol{z}_j) + \delta_{ij}^{(b)} \widehat{\varepsilon}_{ij}$.
 - (ii) Based on $\left\{ (\widetilde{\mathbf{X}}_i, Y_{ij}^{*(b)}) \right\}_{j=1,i=1}^{N,n}$, obtain $\widehat{\beta}_{\ell}^{*(b)}(\boldsymbol{z})$ using (2.5) or (2.8), and calculate $\widehat{\sigma}_{n.\ell\ell}^{*(b)}$ using (3.3) or (3.4).
 - (iii) Construct SCCs for the resampled data $\left\{ (\widetilde{\mathbf{X}}_i, Y_{ij}^{*(b)}) \right\}_{j=1,i=1}^{N,n} : \mathcal{B}_{(b)}^*(\alpha), \ b = 1, \cdots, B,$ $\mathcal{B}_{(b)}^*(\alpha) = \left\{ (\boldsymbol{z}, \boldsymbol{y}) : \boldsymbol{z} \in \Omega, \widehat{\boldsymbol{\beta}}_{\ell}^{*(b)}(\boldsymbol{z}) \widehat{\boldsymbol{\sigma}}_{n,\ell\ell}^{*(b)}(\boldsymbol{z}) Z_{1-\alpha/2} \leq \boldsymbol{y} \leq \widehat{\boldsymbol{\beta}}_{\ell}^{*(b)}(\boldsymbol{z}) + \widehat{\boldsymbol{\sigma}}_{n,\ell\ell}^{*(b)}(\boldsymbol{z}) Z_{1-\alpha/2} \right\}.$
 - (iv) Estimate the coverage rate $\tau_{\ell}(\boldsymbol{z}_{j}, \alpha) = P\{(\boldsymbol{z}_{j}, \widehat{\beta}_{\ell}(\boldsymbol{z}_{j})) \in \mathcal{B}^{*}(\alpha) | \mathbb{X}\}$ using $\widehat{\tau}_{\ell}(\boldsymbol{z}_{j}, \alpha) = \frac{1}{B} \sum_{b=1}^{B} I\{(\boldsymbol{z}_{j}, \widehat{\beta}_{\ell}(\boldsymbol{z}_{j})) \in \mathcal{B}^{*}_{(b)}(\alpha)\}.$
 - (v) Find the root of the equation $\widehat{\tau}_{\ell}(\boldsymbol{z}_{j}, \alpha) = 1 \alpha_{0}$, for j = 1, ..., N, and denote it as $\{\widehat{\alpha}_{\ell}(\boldsymbol{z}_{j}, \alpha_{0})\}_{j=1}^{N}$. The root can be found using the grid method by repeating the last two steps for different values of α .
 - (vi) Take the minimum of $\{\widehat{\alpha}_{\ell}(z_j,\alpha_0)\}_{j=1}^N$ and denote it as $\widehat{\alpha}_{\ell} \equiv \widehat{\alpha}_{\ell}(\alpha_0)$.
- Step 4. Construct the final SCCs: $\mathcal{B}(\widehat{\alpha}_{\ell}) = \{(\boldsymbol{z}, y) : \boldsymbol{z} \in \Omega, \widehat{\beta}_{\ell}(\boldsymbol{z}) \widehat{\sigma}_{n,\ell\ell}(\boldsymbol{z}) Z_{1-\widehat{\alpha}_{\ell}/2} \leq y \leq \widehat{\beta}_{\ell}(\boldsymbol{z}) + \widehat{\sigma}_{n,\ell\ell}(\boldsymbol{z}) Z_{1-\widehat{\alpha}_{\ell}/2} \}.$

4. Implementation

The proposed procedure can be implemented using our R package "FDAimage" (Yu et al., 2019), in which the bivariate spline basis is generated using the R package "BPST" (Wang et al., 2019). When the response imaging seems to be a realization from some smooth function, we suggest using the smoothing parameter r=1 and degree $d \geq 5$, which achieves full estimation power asymptotically (Lai and Schumaker, 2007). In contrast, if there are sharp edges on the images, we suggest considering the PCST presented in Section 2.4.

Selecting suitable values for the smoothing parameters is important to good model fitting. To select $\rho_{n,\ell}$, for $\ell = 0, \ldots, p$, we used K-fold cross-validation (CV). The individuals are randomly partitioned into K groups, where one group is retained as a test set, and the remaining K-1 groups are used as training sets. The CV process is repeated K times (the folds), with each of the K groups used exactly once as the validation data. Then, the K-fold CV score is

$$CV(\rho_{n,0},\ldots,\rho_{n,p}) = K^{-1} \sum_{k=1}^{K} (|\mathcal{V}_k|N)^{-1} \sum_{i \in \mathcal{V}_k} \sum_{j=1}^{N} \{Y_{ij} - \widetilde{\mathbf{X}}_i^{\top} \widehat{\boldsymbol{\beta}}_{-k}(\boldsymbol{z}_j)\}^2,$$

where V_k is the kth testing set for k = 1, ..., K, and $\widehat{\beta}_{-k}$ is the corresponding estimator after removing the kth testing set. We use K = 5 in our numerical examples.

To determine an optimal triangulation, the criterion usually considers the shape, size, or number of triangles. In terms of shape, a "good" triangulation usually refers to one with well-shaped triangles without small angles and/or obtuse angles. Therefore, for a given number of triangles, Lai and Schumaker (2007) and Lindgren et al. (2011) recommended selecting the triangulation according to "max-min" criterion, which maximizes the minimum angle of all the angles of the triangles in the triangulation. With respect to the number of triangles, our numerical studies show that a lower limit of the number of triangles is necessary to capture the features of the images. However, once this minimum number has been reached, refining the triangulation further usually has little effect on the fitting process. In practice, when using higher-order BPST smoothing, we suggest taking the number of triangles as $H_n = \min\{\lfloor c_1 n^{1/(2d+2)} N^{1/2} \rfloor, N/10\}$, where c_1 is a tuning parameter. We find that $c_1 \in [0.3, 2.0]$ works well in our numerical studies. When using the PCST, we suggest taking the number of triangles as $H_n = \min\{\lfloor c_2 n^{-1/4} N \rfloor, N/2\}$, with $c_2 \in [0.3, 2.0]$. Once H_n is chosen, we can build the triangulation using typical triangulation construction methods, such as Delaunay triangulation and DistMesh (Persson and Strang, 2004).

5. Simulation Studies

In this section, we conduct two Monte Carlo simulation studies using our R package "FDAimage" (Yu et al., 2019) to examine the finite-sample performance of the proposed methodology. The triangulations used here can be found in the data set in the "FDAimage" package. To illustrate the performance of our estimation method, we compare the proposed spline method with the kernel method proposed by Zhu et al. (2014) (Kernel) and the tensor regression method of Li and Zhang (2017) (Tensor). To implement the kernel method, we use the R Package SVCM, which is publicly available at https://github.com/BIG-S2/SVCM. For the tensor method, the accompanying MATLAB code at https://ani.stat.fsu.edu/~henry/TensorEnvelopes_html.html is used. We compare the proposed method with the tensor regression approach in Li and Zhang (2017) and the three-stage FDA approach in Zhu et al. (2014).

5.1. Example 1

To illustrate the advantage of the proposed method over a complex domain, we study the horseshoe domain in Sangalli et al. (2013). The response images are generated from the following model: $Y_{ij} = \beta_0^o(z_j) + X_i\beta_1^o(z_j) + \eta_i(z_j) + \sigma \varepsilon_{ij}$, for i = 1, ..., n, j = 1, ..., N, and $z_j \in \Omega$. To understand the advantages and disadvantages of different methods, we consider

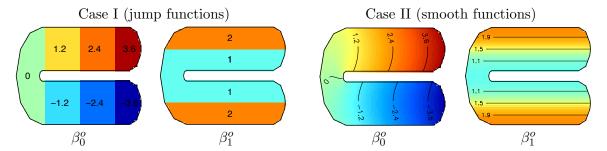


Figure 5.1: The true coefficient functions in Simulation Example 1.

Table 5.1: Estimation errors of the coefficient estimators, $\sigma = 2.0$.

Function	200	Method	$\lambda_1 = 0.0$	$3, \lambda_2 = 0.006$	$\lambda_1 = 0.2, \ \lambda_2 = 0.05$		
Type	n	Method	β_0	β_1	β_0	β_1	
	50	BPST	0.0139	0.0182	0.0145	0.0189	
		PCST	0.0088	0.0090	0.0094	0.0097	
		Kernel	0.0801	0.0819	0.0807	0.0826	
T		Tensor	0.0799	0.0248	0.0799	0.0254	
Jump	100	BPST	0.0090	0.0118	0.0093	0.0122	
		PCST	0.0044	0.0044	0.0047	0.0047	
		Kernel	0.0400	0.0405	0.0403	0.0409	
		Tensor	0.0395	0.0166	0.0399	0.0171	
	50	BPST	0.0026	0.0032	0.0032	0.0041	
		PCST	0.0088	0.0090	0.0119	0.0139	
		Kernel	0.0801	0.0819	0.0807	0.0826	
Smooth		Tensor	0.0799	0.0256	0.0806	0.0271	
	100	BPST	0.0016	0.0019	0.0019	0.0022	
		PCST	0.0070	0.0086	0.0073	0.0090	
		Kernel	0.0400	0.0405	0.0403	0.0409	
		Tensor	0.0399	0.0168	0.0402	0.0179	

two types of coefficient functions in the above image-on-scalar regression model: (I) functions with jumps; and (II) smooth functions. The true coefficient functions are shown in Figure 5.1.

For each image, we set the resolution as 100×50 (pixels). The true signal falls only within the horseshoe domain (3182 pixels); outside the domain is pure noise. We generate the scalar covariate $X_i \sim N(0,1)$, and then truncate it by [-3,+3]. We set $\eta_i(z) = \sum_{k=1}^2 \lambda_k^{1/2} \xi_{ik} \psi_k(z)$, where $(\lambda_1, \lambda_2) = (0.1, 0.02)$ or (0.2, 0.05) and ξ_{i1} and $\xi_{i2} \sim N(0,1)$, $\psi_1(z) = c_1 \sin(2\pi z_1)$, and $\psi_2(z) = c_2 \cos(2\pi z_2)$. Let $c_1 = 0.56$ and $c_2 = 0.61$, such that ψ_1 and ψ_2 are orthonormal functions on Ω . The measurement error ε_{ij} is independently generated from N(0,1) and $\sigma = 1.0, 2.0$.

To fit the model, we consider the BPST and PCST methods presented in Section 2. To obtain the BPST estimators, we set d = 5 and r = 0 when generating the bivariate spline basis functions. Figure B.1 in the Appendix B illustrates the triangulations used for the BPST and

PCST. The triangulation used for the BPST (\triangle_1) contains 90 triangles (73 vertices), and the triangulation used for the PCST (\triangle_2) contains 346 triangles (226 vertices).

We quantify the estimation accuracy of the coefficient functions using the mean squared error (MSE). Table 5.1 provides the average MSE (across 500 Monte Carlo experiments) for two types of coefficient functions. To save space, we present the results for $\sigma = 2.0$ only; the results for $\sigma = 1.0$ are presented in Table B.1 in the Appendix B. As expected, the estimation accuracy of all the methods improves as the sample size increases or the noise level decreases. In both scenarios, the BPST and PCST outperform the other two competitors, reflecting the advantage of our method over a complex domain. When the true coefficient functions are smooth, the BPST provides the best estimation, followed by the PCST. On the other hand, when the true coefficient function contains jumps, the PCST provides a better result. For the tensor regression, the estimator of $\beta_1^o(\cdot)$ is much more accurate than that of $\beta_0^o(\cdot)$, owing to the design of the coefficient function. Figure 5.1 shows that, in contrast to the intercept function of $\beta_0^o(\cdot)$, the true slope function of $\beta_1^o(\cdot)$ is still smooth across the complex boundary. Moreover, when the coefficient function is smooth across the boundary, the estimation accuracy is also affected by the domain of the true signal. The performance of the kernel method is not affected by the design of the coefficient functions. Instead, it depends heavily on the noise level, owing to the three-stage structure.

5.2. Example 2

In this example, we simulate the data by considering the domains of the fifth and 35th slices of the brain images illustrated in Section 6 as the domain Ω . We generate response images based on a set of smooth coefficient functions from the following model: $Y_{ij} = \sum_{\ell=0}^{2} X_{i\ell} \beta_{\ell}^{o}(\boldsymbol{z}_{j}) +$ $\eta_i(z_j) + \sigma \varepsilon_{ij}$, for i = 1, ..., n, j = 1, ..., N, and $z_j \in \Omega$, where $\beta_0^o(z) = 5\{(z_1 - 0.5)^2 + (z_2 - 0.5)^2 + (z_2 - 0.5)^2 + (z_3 - 0.5)^2 +$ $(0.5)^2$, $\beta_1^o(z) = -1.5z_1^3 + 1.5z_2^3$ and $\beta_2^o(z) = 2 - 2\exp[-8\{(z_1 - 0.5)^2 + (z_2 - 0.5)^2\}]$. the true coefficient images are shown in the first columns of Figures B.5 and B.6 in the Appendix B for the fifth and 35th slices, respectively. For each image, we simulate the data at all 79×95 pixels. To mimic real brain images, the true signals are generated only on the pixels/voxels (3476 or 5203 pixels in total) within the brain domain; outside the boundary of the brain, the image contains only noise. We set $X_{i0} = 1$ and generate $\widetilde{\mathbf{X}}_i = (X_{i1}, X_{i2})^{\top} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma} =$ $\begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$ and $X_{i\ell}$ truncated by [-3, +3]. For the error terms, we set $\eta_i(\boldsymbol{z}) = \sum_{k=1}^2 \lambda_k^{1/2} \xi_{ik} \psi_k(\boldsymbol{z})$, where ξ_{i1} and $\xi_{i2} \sim N(0,1)$, $\psi_1(z) = 1.488\{\sin(\pi z_1) - 1.5\}$, $\psi_2(z) = 1.939\cos(2\pi z_2)$, and $(\lambda_1, \lambda_2) = (0.1, 0.02)$ or (0.2, 0.05). The measurement error ε_{ij} is independently generated from N(0,1) and $\sigma=0.5, 1.0$. To conserve space, we show only the results for the domain of the fifth slice for $\sigma = 1.0$ here. The results for $\sigma = 0.5$ and those based on the domain of the 35th slice are shown in Appendix B.

Table 5.2: Estimation errors of the coefficient function estimators, $\sigma = 1.0$.

\overline{n}	Method	$\lambda_1 = 0$	$0.1, \ \lambda_2 =$	= 0.02	$\lambda_1 = 0.2, \ \lambda_2 = 0.05$		
		β_0	β_1	β_2	β_0	β_3	β_2
50	$BPST(\triangle_3)$	0.003	0.005	0.005	0.007	0.011	0.010
	$BPST(\triangle_4)$	0.003	0.005	0.005	0.007	0.010	0.009
	Kernel	0.023	0.032	0.032	0.026	0.037	0.037
	Tensor	0.023	0.013	0.019	0.026	0.017	0.024
100	$BPST(\Delta_3)$	0.002	0.002	0.002	0.003	0.005	0.005
	$BPST(\triangle_4)$	0.002	0.002	0.002	0.003	0.004	0.004
	Kernel	0.011	0.015	0.015	0.013	0.018	0.018
	Tensor	0.011	0.007	0.011	0.013	0.009	0.013

Because the functions in this example are smooth, for the bivariate spline approach, we consider only the BPST method. To further study the effect of different triangulations, we consider \triangle_3 and \triangle_4 ; see Figure B.4 in the Appendix B. Similarly to Section 5.1, we summarize the MSE for different coefficient functions based on 500 Monte Carlo experiments in Table 5.2. Columns 2–5 in Figure B.5 in the Appendix B show the estimated coefficient functions using the kernel, tensor and BPST methods, respectively. Table 5.2 and Figure B.5 in the Appendix B show that the estimation accuracy improves for all methods as the sample size increases or the noise level decreases. In all settings, the BPST method has the smallest MSE compared with the kernel and tensor methods, reflecting the advantage of our method in estimating the coefficient functions and, hence, the regression function. Because the kernel and tensor methods are both designed for a rectangle domain, the estimation accuracy can be affected by the noise outside the domain. Futhermore, the MSE is invariable across two triangulations, thus, \triangle_3 might be sufficient to capture the feature in the data set. This also implies that when this minimum number of triangles is reached, further refining the triangulation has little effect on the fitting process, but makes the computational burden unnecessarily heavy.

Finally, we illustrate the finite-sample performance of the proposed SCCs for the coefficient functions described in Section 3. In particular, we report the empirical coverage probabilities of the nominal 95% SCCs using triangulation \triangle_3 . We evaluate the coverage of the proposed SCCs over all pixels on the interior of Ω , and test whether the true functions are entirely covered by the SCCs at these pixels. Table 5.3 summarizes the empirical coverage rate (ECR) for 500 Monte Carlo experiments of the 95% SCCs and the average width of the SCCs. The results clearly show that the ECRs of the SCCs are well approximated to 95%, particularly as the sample size increases. Table 5.3 also reveals that the SCCs tend to be narrower when the sample size becomes larger or the noise level decreases.

6. ADNI Data Analysis

Table 5.3: The coverage rate of the 95% SCCs for the coefficient functions.

			Coverage			Width		
n	λ	σ	β_0	β_1	β_2	β_0	β_1	β_2
50 —	(0.1,0.02)	0.5	0.976	0.928	0.938	0.332	0.362	0.377
	(0.1, 0.02)	1.0	0.976	0.940	0.952	0.358	0.392	0.413
	(0.2,0.05)	0.5	0.962	0.918	0.932	0.445	0.497	0.513
		1.0	0.970	0.930	0.940	0.478	0.527	0.544
	(0.1, 0.02)	0.5	0.970	0.956	0.956	0.234	0.250	0.267
		1.0	0.978	0.968	0.978	0.262	0.285	0.297
100	(0.2,0.05)	0.5	0.956	0.958	0.936	0.313	0.348	0.357
		1.0	0.966	0.964	0.954	0.344	0.378	0.389

To illustrate the proposed method, we consider the spatially normalized FDG (fludeoxyglucose) PET data of the Alzheimer's Disease Neuroimaging Initiative (ADNI). As pointed out in Marcus et al. (2014), FDG-PET images have been shown to be a promising modality for detecting functional brain changes in Alzheimer's Disease (AD). The data can be obtained from the ADNI database at http://adni.loni.usc.edu/. The database contains spatially normalized PET images of 447 subjects. Of these 447 subjects, 112 have normal cognitive functions, considered to be the control group, 213 are diagnosed as mild cognitive impairment (MCI), and 122 are diagnosed as AD. Table B.5 in the Appendix B summarizes the distribution of patients by diagnosis status and sex.

In this study, we examine several patient-level features: (i) demographical features, such as age (Age) and sex (Sex); (ii) a dummy variable for the abnormal diagnosis status "MA" (1 = "AD" or "MCI", zero otherwise); (iii) a dummy variable for "AD" (1 = "AD," zero otherwise); and (iv) dummy variables for the APOE genotype, the strongest genetic risk factor for "AD"; see Corder et al. (1993). We code APOE₁ as a dummy variable for subjects with one epsilon 4 allele, and APOE₂ as subjects who have two alleles.

Noting that the PET images are 3D, we select the 5th, 8th, 15th, 35th, 55th, 62nd, and 6fifth horizontal slices (bottom to up) of the brain from a total of 68 slices to illustrate our method. Each slice of the image contains 79×95 pixels, but the domains of different brain slices are quite different. Specifically, the domain boundary for the bottom slices and upper slices are much more complex than the slices in the middle; more examples can be found in Figure B.7 in the Appendix B. For each slice, we consider the following image-on-scalar regression:

$$Y_i(\boldsymbol{z}_j) = \beta_0(\boldsymbol{z}_j) + \beta_1(\boldsymbol{z}_j) \mathrm{MA}_i + \beta_2(\boldsymbol{z}_j) \mathrm{AD}_i + \beta_3(\boldsymbol{z}_j) \mathrm{Age}_i + \beta_4(\boldsymbol{z}_j) \mathrm{Sex}_i$$
$$+ \beta_5(\boldsymbol{z}_i) \mathrm{APOE}_{1i} + \beta_6(\boldsymbol{z}_i) \mathrm{APOE}_{2i} + \eta_i(\boldsymbol{z}_i) + \sigma(\boldsymbol{z}_i) \varepsilon_i(\boldsymbol{z}_i), \ i = 1, \dots, n.$$

We fit the above model using the BPST method for each slice; see Figure B.7 in the Appendix B for the set of triangulations used for the BPST method. The image maps in Figure

Table 5.1: 10-fold CV results for the ADNI dataset. $(\times 10^{-2})$

Method	Slice 5	Slice 8	Slice 15	Slice 35	Slice 55	Slice 62	Slice 65
BPST	1.4508	1.4809	1.5013	1.5633	2.0693	2.3020	2.6239
Kernel	1.4533	1.4828	1.5021	1.5638	2.0715	2.3060	2.6303
Tensor	1.5010	1.5260	1.5400	1.5900	2.1000	2.3340	2.6400

5.1 and Figures B.8 and B.9 in the Appendix B present the estimated coefficient functions using the BPST (d=5, r=1) method. To evaluate the predictive performance, Table 5.1 reports the 10-fold CV (parts of the images are left out as training sets) MSPE results for the BPST method, kernel method in Zhu et al. (2014), and tensor regression method in Li and Zhang (2017). The table shows that the MSPEs of the BPST method are uniformly smaller than those of the kernel method and tensor regression methods.

Next, we construct the 95% SCCs to check whether the covariates are significant. The yellow and blue colors on the "significance" map in Figure 5.1 indicate the regions in which zero is below the lower SCC or above the upper SCC, respectively. Using these estimated coefficient functions and the 95% SCCs, we can assess the impact of the covariates on the response images. Taking the fifth slice as an example, the main impact of "AD" on in the PET images is an increase in activity in the cerebellum compared with a normal individual. The cerebellum obtains information from the sensory systems, spinal cord, and other parts of the brain, and then regulates motor movements, resulting in smooth and balanced muscular activities. The significance map of "Age" also shows an increase in activity in the cerebellum, and "Sex" shows different effects in the male and female brain images. The significance maps of the covariates for all other slices of the PET image are shown in Figures B.10 – B.11 in the Appendix B. From these figures, we can see that the effect of the covariates on the brain activity level varies between slices, depending on the location of the slice; see the Appendix B for further details.

7. Conclusion

We examine a class of image-on-scalar regression models to efficiently explore the spatial nonstationarity of a regression relationship between imaging responses and scalar predictors, allowing the regression coefficients to change with the pixels. We have proposed an efficient estimation procedure to carry out statistical inference. We have developed a fast and accurate method for estimating the coefficient images, while consistently estimating their standard deviation images. Our method provides coefficient maps and significance maps that highlight and visualize the associations with brain and the potential risk factors, adjusted for other patient-level features, as well as permitting inference. In addition, it allows an easy implementation

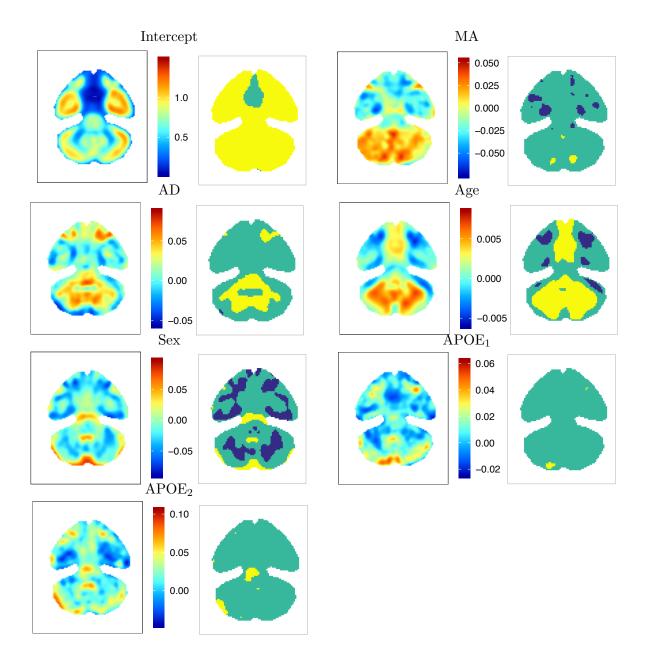


Figure 5.1: The BPST estimate and significance map of the coefficient functions for the fifth slice of the PET images. The yellow and blue colors in the significance map indicate the regions in which zero is below the lower SCC or above the upper SCC, respectively.

of piecewise polynomial representations of various degrees and smoothness over an arbitrary triangulation, and therefore can handle irregular-shaped 2D objects with different visual qualities. This provides enormous flexibility, accommodating various types of nonstationarity that are commonly encountered in imaging data analysis. Our methodology is extendable to 3D images to fully realize its potential usefulness in biomedical imaging. Instead of using bivariate splines over triangulation, the trivariate splines over tetrahedral partitions introduced in Lai and Schumaker (2007) could be well suited, because they have many properties in common with the bivariate splines over triangulation. However, this is a nontrivial task, because the computation is much more challenging for high-resolution 3D images than it is for 2D images, and thus warrants further investigation.

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Appendix A

In the following, we use c, C, c_1, c_2, C_1, C_2 , etc. as generic constants, which may be different even in the same line. For any sequence a_n and b_n , we write $a_n \times b_n$ if there exist two positive constants c_1, c_2 such that $c_1|a_n| \leq |b_n| \leq c_2|a_n|$, for all $n \geq 1$. For a real valued vector \boldsymbol{a} , denote $\|\boldsymbol{a}\|$ its Euclidean norm. For a matrix $\boldsymbol{A} = (a_{ij})$, denote $\|\boldsymbol{A}\|_{\infty} = \max_{i,j} |a_{ij}|$. For any positive definite matrix \boldsymbol{A} , let $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$ be the smallest and largest eigenvalues of \boldsymbol{A} . For a vector valued function $\boldsymbol{g} = (g_0, \ldots, g_p)^{\top}$, denote $\|\boldsymbol{g}\|_{L_2(\Omega)} = \{\sum_{\ell=0}^p \|g_{\ell}\|_{L_2(\Omega)}^2\}^{1/2}$ and $\|\boldsymbol{g}\|_{\infty,\Omega} = \max_{0\leq \ell\leq p} \|g_{\ell}\|_{\infty,\Omega}$, where $\|g_{\ell}\|_{L_2,\Omega}$ and $\|g_{\ell}\|_{\infty,\Omega}$ are the L_2 norm and supremum norm of g_{ℓ} defined at the beginning of Section 2.2. Further denote $\|\boldsymbol{g}\|_{v,\infty,\Omega} = \max_{0\leq \ell\leq p} |g_{\ell}|_{v,\infty,\Omega}$, where $|g_{\ell}|_{v,\infty,\Omega} = \max_{i+j=v} \|\nabla_{z_1}^i \nabla_{z_2}^j g_{\ell}(\boldsymbol{z})\|_{\infty,\Omega}$. For notation simplicity, we drop the subscript Ω in the rest of the paper. For $\boldsymbol{g}^{(1)}(\boldsymbol{z}) = (g_0^{(1)}(\boldsymbol{z}), \ldots, g_p^{(1)}(\boldsymbol{z}))^{\top}$ and $\boldsymbol{g}^{(2)}(\boldsymbol{z}) = (g_0^{(2)}(\boldsymbol{z}), \ldots, g_p^{(2)}(\boldsymbol{z}))^{\top}$, define the empirical inner product as

$$\langle \boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)} \rangle_{n,N} = \frac{1}{nN} \sum_{\ell,\ell'=0}^{p} \sum_{i=1}^{n} \sum_{j=1}^{N} X_{i\ell} X_{i\ell'} g_{\ell}^{(1)}(\boldsymbol{z}_j) g_{\ell'}^{(2)}(\boldsymbol{z}_j),$$
 (A.1)

and the theoretical inner product as

$$\langle \boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)} \rangle = \sum_{\ell,\ell'=0}^{p} E(X_{\ell} X_{\ell'}) \int_{\Omega} g_{\ell}^{(1)}(\boldsymbol{z}) g_{\ell'}^{(2)}(\boldsymbol{z}) d\boldsymbol{z}, \tag{A.2}$$

and denote the corresponding empirical and theoretical norms $\|\cdot\|_{n,N}$ and $\|\cdot\|$.

Furthermore, let $\|\cdot\|_{\mathcal{E}}$ be the norm introduced by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, where, for $\boldsymbol{g}^{(1)}(\boldsymbol{z})$ and $\boldsymbol{g}^{(2)}(\boldsymbol{z})$,

$$\langle \boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)} \rangle_{\mathcal{E}} = \sum_{\ell,\ell'=0}^{p} \int_{\Omega} \left\{ \sum_{i+j=2} {2 \choose i} (\nabla_{z_1}^{i} \nabla_{z_2}^{j} g_{\ell}^{(1)}) \right\} \left\{ \sum_{i+j=2} {2 \choose i} (\nabla_{z_1}^{i} \nabla_{z_2}^{j} g_{\ell'}^{(2)}) \right\} dz_1 dz_2.$$

Let $A(\Omega)$ be the area of the domain Ω , and without loss of generality, we assume $A(\Omega) = 1$ in the rest of the article. Note that the triangulation for different coefficient function can be different from each other. For notational convenience in the proof below, we consider a common triangulation for all the explanatory variables: $\mathbf{B}_0(z) = \mathbf{B}_1(z) = \cdots = \mathbf{B}_p(z) = \mathbf{B}(z)$, and $\beta_\ell(z_j) = \mathbf{B}^\top(z_j) \gamma_\ell$.

A.1. Properties of bivariate splines

We cite two important results from Lai and Schumaker (2007).

Lemma A.1 (Theorem 2.7, Lai and Schumaker (2007)). Let $\{B_m\}_{m\in\mathcal{M}}$ be the Bernstein polynomial basis for spline space $\mathcal{S}_d^r(\triangle)$ defined over a π -quasi-uniform triangulation \triangle . Then there exist positive constants c, C depending on the smoothness r, d, and the shape parameter π such that $c|\triangle|^2 \sum_{m\in\mathcal{M}} \gamma_m^2 \leq \|\sum_{m\in\mathcal{M}} \gamma_m B_m\|_{L_2}^2 \leq C|\triangle|^2 \sum_{m\in\mathcal{M}} \gamma_m^2$.

Lemma A.2 (Theorems 10.2 and 10.10, Lai and Schumaker (2007)). Suppose that $|\Delta|$ is a π -quasi-uniform triangulation of a polygonal domian Ω , and $g(\cdot) \in \mathcal{W}^{d+1,\infty}(\Omega)$.

- (i) For bi-integer (a_1, a_2) with $0 \le a_1 + a_2 \le d$, there exists a spline $g^*(\cdot) \in \mathcal{S}_d^0(\triangle)$ such that $\|\nabla_{z_1}^{a_1}\nabla_{z_2}^{a_2}(g-g^*)\|_{\infty} \le C|\triangle|^{d+1-a_1-a_2}|g|_{d+1,\infty}$, where C is a constant depending on d, and the shape parameter π .
- (ii) For bi-integer (a_1, a_2) with $0 \le a_1 + a_2 \le d$, there exists a spline $g^{**}(\cdot) \in \mathcal{S}_d^r(\triangle)$ $(d \ge 3r + 2)$ such that $\|\nabla_{z_1}^{a_1}\nabla_{z_2}^{a_2}(g g^{**})\|_{\infty} \le C|\triangle|^{d+1-a_1-a_2}|g|_{d+1,\infty}$, where C is a constant depending on d, r, and the shape parameter π .

Lemma A.2 shows that $\mathcal{S}_d^0(\Delta)$ has full approximation power, and $\mathcal{S}_d^r(\Delta)$ also has full approximation power if $d \geq 3r + 2$. For any $g(\cdot)$ in Sobolev space $\mathcal{C}^{(0)}(\Omega)$, there exists a spline $g^*(\cdot) \in \mathcal{PC}(\Delta)$ such that $\|g - g^*\|_{\infty} \leq C|\Delta| \|g\|_{\infty}$.

Lemma A.3. Let $\mathbf{g}(\mathbf{z}) = (g_0(\mathbf{z}), \dots, g_p(\mathbf{z}))^{\top}$, where $g_{\ell}(\mathbf{z}) = \sum_{m \in \mathcal{M}} \gamma_{\ell m} B_m(\mathbf{z})$. Then, under Assumptions (A3) and (A5), $\|\mathbf{g}\| \times \sum_{\ell=0}^p \|g_{\ell}\|_{L_2}$.

Proof. By (A.1), $\|\mathbf{g}\|^2 = \sum_{\ell,\ell'=0}^p E(X_\ell X_{\ell'}) \int_{\Omega} g_\ell(\mathbf{z}) g_{\ell'}(\mathbf{z}) d\mathbf{z} = \int_{\Omega} \mathbf{g}^\top(\mathbf{z}) \mathbf{\Sigma}_X \mathbf{g}(\mathbf{z}) d\mathbf{z}$. According to Assumptions (A3) and (A5), $\|\mathbf{g}\|^2 \asymp \int_{\Omega} \mathbf{g}^\top(\mathbf{z}) \mathbf{g}(\mathbf{z}) d\mathbf{z} \asymp \sum_{\ell=0}^p \|g_\ell\|_{L_2}$.

Lemma A.4. Under Assumptions (A4) and (A5), for any Bernstein basis polynomials $B_m(z)$, $m \in \mathcal{M}$, of degree $d \geq 0$, we have

$$\max_{m \in \mathcal{M}} \left| \frac{1}{N} \sum_{j=1}^{N} B_m^k(\boldsymbol{z}_j) - \int_{\Omega} B_m^k(\boldsymbol{z}) d\boldsymbol{z} \right| = O\left(|\triangle| N^{-1/2}\right), \ 1 \le k < \infty, \tag{A.3}$$

$$\max_{m,m'\in\mathcal{M}} \left| \frac{1}{N} \sum_{j=1}^{N} B_m(\boldsymbol{z}_j) B_{m'}(\boldsymbol{z}_j) - \int_{\Omega} B_m(\boldsymbol{z}) B_{m'}(\boldsymbol{z}) d\boldsymbol{z} \right| = O\left(|\triangle| N^{-1/2}\right), \tag{A.4}$$

$$\max_{m,m'\in\mathcal{M}} \left| \frac{1}{N^2} \sum_{j,j'=1}^{N} G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) B_m(\boldsymbol{z}_j) B_{m'}(\boldsymbol{z}_{j'}) - \int_{\Omega^2} G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_m(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') d\boldsymbol{z} d\boldsymbol{z}' \right|$$

$$= O\left(N^{-1/2} |\Delta|^3\right), \tag{A.5}$$

$$\max_{m \in \mathcal{M}} \left| \|\sigma B_m\|_{N, L_2}^2 - \|\sigma B_m\|_{L_2}^2 \right| = \max_{m \in \mathcal{M}} \left| \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \sigma^2(\boldsymbol{z}_j) - \int_{\Omega} \sigma^2(\boldsymbol{z}) B_m^2(\boldsymbol{z}) d\boldsymbol{z} \right| \\
= O\left(N^{-1/2} |\Delta|\right). \tag{A.6}$$

Proof. Note that there are $d^* = (d+1)(d+2)/2$ Bernstein basis polynomials on each triangle and $\int_{\Omega} B_m^k(\boldsymbol{z}) d\boldsymbol{z} = \int_{T_{\lceil m/d^* \rceil}} B_m^k(\boldsymbol{z}) d\boldsymbol{z}$, for any $k \geq 1$. For piecewise constant basis functions, we have $B_m(\boldsymbol{z}) = I(\boldsymbol{z} \in T_m)$, then

$$\left| \frac{1}{N} \sum_{j=1}^{N} B_m^k(\boldsymbol{z}_j) - \int_{\Omega} B_m^k(\boldsymbol{z}) d\boldsymbol{z} \right| = \left| \frac{1}{N} \sum_{j=1}^{N} I(\boldsymbol{z}_j \in T_m) - A(T_m) \right|.$$

According to Assumption (A5),

$$\max_{m \in \mathcal{M}} \left| \frac{1}{N} \sum_{j=1}^{N} B_m^k(\boldsymbol{z}_j) - \int_{\Omega} B_m^k(\boldsymbol{z}) d\boldsymbol{z} \right| \leq C N^{-1/2} |\Delta|.$$

For any j = 1, ..., N, let \mathcal{V}_j be the jth pixel, and it is clear that

$$\left|\frac{1}{N}\sum_{j=1}^N B_m^k(\boldsymbol{z}_j) - \int_{\Omega} B_m^k(\boldsymbol{z}) d\boldsymbol{z}\right| \leq \left|\sum_{j=1}^N \int_{\mathcal{V}_j} \{B_m^k(\boldsymbol{z}_j) - B_m^k(\boldsymbol{z})\} d\boldsymbol{z}\right| + \int_{\Omega \setminus \cup \mathcal{V}_j} B_m^k(\boldsymbol{z}) d\boldsymbol{z}.$$

If $d \geq 1$, by the properties of bivariate spline basis functions in Lai and Schumaker (2007), $\int_{\Omega \setminus \cup \mathcal{V}_j} B_m^k(\boldsymbol{z}) d\boldsymbol{z} = O(N^{-1/2}|\triangle|),$ and

$$\left| \sum_{j=1}^{N} \int_{\mathcal{V}_{j}} \{B_{m}^{k}(\boldsymbol{z}_{j}) - B_{m}^{k}(\boldsymbol{z})\} d\boldsymbol{z} \right| \leq \sum_{\{j: \boldsymbol{z}_{j} \in T_{\lceil m/d^{*} \rceil}\}} \int_{\mathcal{V}_{j}} |B_{m}^{k}(\boldsymbol{z}_{j}) - B_{m}^{k}(\boldsymbol{z})| d\boldsymbol{z}$$
$$\leq C(N|\Delta|^{2}) \times N^{-1} \times (N^{-1/2}|\Delta|^{-1}) \leq CN^{-1/2}|\Delta|.$$

Thus, (A.3) holds. The proof of (A.4) is similar to the proof (A.3), thus omitted. Next, for any $m, m' \in \mathcal{M}$,

$$\frac{1}{N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j'}) - \int_{\Omega^{2}} G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') d\boldsymbol{z} d\boldsymbol{z}'$$

$$= \sum_{j=1}^{N} \sum_{j'=1}^{N} \int_{\mathcal{V}_{j} \times \mathcal{V}_{j'}} \left\{ G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j'}) - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') \right\} d\boldsymbol{z} d\boldsymbol{z}'$$

$$+ \int_{\Omega^{2} \setminus \bigcup_{i,j'} \mathcal{V}_{j} \times \mathcal{V}_{j'}} \left\{ G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j'}) - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') \right\} d\boldsymbol{z} d\boldsymbol{z}'.$$

As $N \to \infty$,

$$\int_{\Omega^2 \setminus \bigcup_{j,j'} \mathcal{V}_j \times \mathcal{V}_{j'}} \{ G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) B_m(\boldsymbol{z}_j) B_{m'}(\boldsymbol{z}_{j'}) - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_m(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') \} \, d\boldsymbol{z} d\boldsymbol{z}' = O\left(\frac{|\triangle|^3}{\sqrt{N}}\right).$$

Notice that

$$\left| \sum_{j=1}^{N} \sum_{j'=1}^{N} \int_{\mathcal{V}_{j} \times \mathcal{V}_{j'}} \left\{ G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j'}) - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') \right\} d\boldsymbol{z} d\boldsymbol{z}' \right|$$

$$\leq \sum_{\{(j,j'): \boldsymbol{z}_{j} \in T_{\lceil m/d^{*} \rceil}, \boldsymbol{z}_{j'} \in T_{\lceil m/d^{*} \rceil}\}} \int_{\mathcal{V}_{j} \times \mathcal{V}_{j'}} \omega_{jj'} (G_{\eta} K_{m}, 2N^{-1/2}) d\boldsymbol{z} d\boldsymbol{z}',$$

where $K_m(\boldsymbol{z}, \boldsymbol{z}') = B_m(\boldsymbol{z}) B_m(\boldsymbol{z}')$ and

$$\omega_{jj'}(g,\varrho) = \sup_{\substack{(\boldsymbol{z}_1,\boldsymbol{z}_1'),(\boldsymbol{z}_2,\boldsymbol{z}_2') \in \mathcal{V}_j \times \mathcal{V}_{j'}, \\ \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|^2 + \|\boldsymbol{z}_1' - \boldsymbol{z}_2'\|^2 = \varrho^2}} |g(\boldsymbol{z}_1,\boldsymbol{z}_1') - g(\boldsymbol{z}_2,\boldsymbol{z}_2')|$$

is the modulus of continuity of g on $\mathcal{V}_j \times \mathcal{V}_{j'}$. Therefore, by Assumption (A4), we have

$$\left| \sum_{j=1}^{N} \sum_{j'=1}^{N} \int_{\mathcal{V}_{j} \times \mathcal{V}_{j'}} \left\{ G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j'}) - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}') \right\} d\boldsymbol{z} d\boldsymbol{z}' \right|$$

$$\leq (N|\Delta|^{2})^{2} \times N^{-2} \times (N^{-1/2}|\Delta|^{-1}) = O(N^{-1/2}|\Delta|^{3}).$$

Thus, (A.5) follows.

Finally, note that

$$\left| \frac{1}{N} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) \sigma^2(\boldsymbol{z}_j) - \int_{\Omega} \sigma^2(\boldsymbol{z}) d\boldsymbol{z} \right| \leq \left| \sum_{j=1}^{N} \int_{\mathcal{V}_j} \{B_m^2(\boldsymbol{z}_j) \sigma^2(\boldsymbol{z}_j) - B_m^2(\boldsymbol{z}) \sigma^2(\boldsymbol{z}) \} d\boldsymbol{z} \right| + \int_{\Omega \setminus \cup \mathcal{V}_j} |B_m^2(\boldsymbol{z}_j) \sigma^2(\boldsymbol{z}_j) - B_m^2(\boldsymbol{z}) \sigma^2(\boldsymbol{z}) | d\boldsymbol{z}.$$

It is easy to see that $\int_{\Omega\setminus \cup \mathcal{V}_j} |B_m^2(\boldsymbol{z}_j)\sigma^2(\boldsymbol{z}_j) - B_m^2(\boldsymbol{z})\sigma^2(\boldsymbol{z})|d\boldsymbol{z} = O(N^{-1/2}|\Delta|)$. Denote $\omega_j(g,\varrho) = \sup_{\boldsymbol{z},\boldsymbol{z}'\in\mathcal{V}_j,\|\boldsymbol{z}-\boldsymbol{z}'\|=\varrho} |g(\boldsymbol{z})-g(\boldsymbol{z}')|$ is the modulus of continuity of g on the jth pixel \mathcal{V}_j , then by Assumption (A4), we have

$$\left| \sum_{j=1}^{N} \int_{\mathcal{V}_{j}} \{B_{m}^{2}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) - B_{m}^{2}(\boldsymbol{z}) \sigma^{2}(\boldsymbol{z})\} d\boldsymbol{z} \right| \leq \sum_{\{j: \boldsymbol{z}_{j} \in T_{\lceil m/d^{*} \rceil}\}} \int_{\mathcal{V}_{j}} \omega_{j} (B_{m}^{2} \sigma^{2}, 2N^{-1/2}) d\boldsymbol{z}$$
$$\leq C(N|\Delta|^{2}) \times N^{-1} \times (N^{-1/2}|\Delta|^{-1}) \leq CN^{-1/2}|\Delta|.$$

We obtain (A.6).

Lemma A.5. For any $m \in \mathcal{M}$, $0 \leq \ell, \ell' \leq p$, let $\Phi_{m,\ell,\ell'} = E(X_{\ell}X_{\ell'}) \int_{\Omega} B_m^2(\boldsymbol{z}) d\boldsymbol{z}$. Suppose Assumptions (A3) and (A5) hold, and $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, then with probability 1, one has

$$\max_{m \in \mathcal{M}} \max_{0 \le \ell, \ell' \le p} \left| \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) X_{i\ell} X_{i\ell'} - \Phi_{m,\ell,\ell'} \right| = O\left\{ n^{-1/2} |\Delta|^2 (\log n)^{1/2} + N^{-1/2} |\Delta| \right\}.$$

Proof. Let $\zeta_{i,m} \equiv \zeta_{i,m,\ell,\ell'} = \frac{1}{N} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) X_{i\ell} X_{i\ell'}$. If $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, then by (A.3), we can show that $E(\zeta_{i,m}) = \frac{1}{N} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) E(X_{\ell} X_{\ell'}) \simeq |\Delta|^2$, and $E(\zeta_{i,m})^2 = \left\{ \frac{1}{N} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) \right\}^2 E(X_{\ell} X_{\ell'})^2 \simeq |\Delta|^4$.

Next define a sequence $D_n = n^{\alpha}$ with $\alpha \in (1/3, 1/2)$. We make use of the following truncated and tail decomposition $X_{i\ell\ell'} = X_{i\ell}X_{i\ell'} = X_{i\ell\ell',1}^{D_n} + X_{i\ell\ell',2}^{D_n}$, where $X_{i\ell\ell',1}^{D_n} = X_{i\ell}X_{i\ell'}I\{|E(X_{i\ell}X_{i\ell'}| > D_n\}, X_{i\ell\ell',2}^{D_n} = X_{i\ell}X_{i\ell'}I\{|X_{i\ell}X_{i\ell'}| \leq D_n\}$. Correspondingly the truncated and tail parts of $\varsigma_{i,m}$ are $\varsigma_{i,m,v} \equiv \varsigma_{i,m,v,\ell,\ell'} = \frac{1}{N} \sum_{j=1}^{N} B_m^2(\mathbf{z}_j) X_{i\ell\ell',v}^{D_n}$, v = 1, 2. According to Assumption (A3), for any $\ell, \ell' = 0, \ldots, p$,

$$\sum_{n=1}^{\infty} P\left\{ |X_{n\ell} X_{n\ell'}| > D_n \right\} \le \sum_{n=1}^{\infty} \frac{E \left| X_{n\ell} X_{n\ell'} \right|^3}{D_n^3} \le C_b \sum_{n=1}^{\infty} D_n^{-3} < \infty.$$

By Borel-Cantelli Lemma, $\frac{1}{N} \sum_{j=1}^{N} B_m^2(\boldsymbol{z}_j) X_{i\ell\ell',1}^{D_n} = 0$, almost surely. So for any $k \geq 1$, $\sup_{m,\ell,\ell'} |n^{-1} \sum_{i=1}^{n} \varsigma_{i,m,1}| = O_{a.s.}(n^{-k})$. Since $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$,

$$\begin{split} |E(\varsigma_{i,m,1})| &= |E(X_{i\ell\ell',1}^{D_n})| \left\{ \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \right\} \\ &\leq D_n^{-2} E \left| X_{i\ell} X_{i\ell'} \right|^3 \left\{ \int_{\Omega} B_m^2(\boldsymbol{z}) d\boldsymbol{z} + O(N^{-1/2} |\Delta|) \right\} \leq C D_n^{-2} |\Delta|^2. \end{split}$$

Next, we consider the truncated part $\zeta_{i,m,2}$. Define $\zeta_{i,m,2}^* = \zeta_{i,m,2} - E(\zeta_{i,m,2})$, then $E\zeta_{i,m,2}^* = 0$, and

$$E(\varsigma_{i,m,2}^*)^2 = E(\varsigma_{i,m,2})^2 - (E\varsigma_{i,m,2})^2 = \left\{\frac{1}{N}\sum_{j=1}^N B_m^2(\boldsymbol{z}_j)\right\}^2 \left\{E(X_{i\ell\ell',2}^{D_n})^2 - (EX_{i\ell\ell',2}^{D_n})^2\right\}.$$

Note that $E(X_{i\ell\ell',1}^{D_n})^2 \leq D_n^{-1}E \left| X_{i\ell}X_{i\ell'} \right|^3 \leq cD_n^{-1}$, thus, $E(X_{i\ell\ell',2}^{D_n})^2 = E(X_{i\ell\ell'})^2 - E(X_{i\ell\ell',1}^{D_n})^2 = E(X_{i\ell\ell',1})^2 - o(1)$. Therefore, there exists c_{ς} such that for large n, we have $E(\varsigma_{i,m,2}^*)^2 \geq c_{\varsigma}E(X_{i\ell\ell'})^2 \times \left\{ \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \right\}^2$. Next for any k > 2,

$$E \left| \varsigma_{i,m,2}^* \right|^k = E \left| \varsigma_{i,m,2} - E \left(\varsigma_{i,m,2} \right) \right|^k \le 2^{k-1} \left(E \left| \varsigma_{i,m,2} \right|^k + \left| E \left(\varsigma_{i,m,2} \right) \right|^k \right)$$
$$= 2^{k-1} \left\{ E \left| X_{i\ell\ell',2}^{D_n} \right|^k + O(1) \right\} \left\{ \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \right\}^k,$$

then there exists $C_{\varsigma} > 0$ such that for any k > 2 and large n,

$$E \left| \varsigma_{i,m,2}^* \right|^k \le 2^{k-1} \left\{ D_n^{k-2} E \left(X_{i\ell\ell'} \right)^2 + O(1) \right\} \left\{ \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \right\}^k$$

$$\le 2^k D_n^{k-2} E \left(\varsigma_{i,m,2}^* \right)^2 \left\{ \frac{1}{N} \sum_{j=1}^N B_m^2(\boldsymbol{z}_j) \right\}^{k-2} \le \left(C_{\varsigma} D_n |\Delta|^2 \right)^{k-2} k! E \left(\varsigma_{i,m,2}^* \right)^2,$$

which implies that $\{\varsigma_{i,m,2}^*\}_{i=1}^n$ satisfies Cramér's condition with constant $C_{\varsigma}D_n|\Delta|^2$. Applying Bernstein's inequality to $\sum_{i=1}^n \varsigma_{i,m,2}^*$, for k>2 and any large enough $\delta>0$,

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\varsigma_{i,m,2}^{*}\right| \geq \delta n^{-1/2}|\Delta|^{2}(\log n)^{1/2}\right\} \leq 2\exp\left\{-\frac{\delta^{2}\log(n)}{4 + 2C_{\varsigma}D_{n}\delta(\log n)^{1/2}n^{-1/2}}\right\}.$$

Assume that $|\Delta|^{-2} \asymp n^{\tau}$ for some $0 < \tau < \infty$, we have

$$\sum_{n=1}^{\infty} P\left\{ \max_{\substack{m \in \mathcal{M} \\ 0 \le \ell, \ell' \le p}} \left| \frac{1}{n} \sum_{i=1}^{n} \varsigma_{i,m,2}^* \right| \ge \delta n^{-1/2} |\triangle|^2 (\log n)^{1/2} \right\} \le 2 \sum_{n=1}^{\infty} \sum_{\substack{m \in \mathcal{M} \\ 0 \le \ell, \ell' \le p}} \sum_{n=1}^{\infty} n^{-2-\tau} < \infty.$$

Thus, $\sup_{m,\ell,\ell'} \left| n^{-1} \sum_{i=1}^n \varsigma_{i,m,2}^* \right| = O_{a.s.} \left\{ n^{-1/2} |\triangle|^2 (\log n)^{1/2} \right\}$ as $n \to \infty$, by Borel-Cantelli Lemma. Furthermore,

$$\max_{m,\ell,\ell'} \left| n^{-1} \sum_{i=1}^{n} \varsigma_{i,m} - E \varsigma_{i,m} \right| \leq \max_{m,\ell,\ell'} \left| n^{-1} \sum_{i=1}^{n} \varsigma_{i,m,1} \right| + \max_{m,\ell,\ell'} \left| n^{-1} \sum_{i=1}^{n} \varsigma_{i,m,2}^{*} \right| + \max_{m,\ell,\ell'} |E \varsigma_{i,m,1}|
= O_{a.s.}(n^{-k}) + O_{a.s.} \left\{ n^{-1/2} |\Delta|^{2} (\log n)^{1/2} \right\} + O\left(D_{n}^{-2} |\Delta|^{2}\right) = O_{a.s.} \left\{ n^{-1/2} |\Delta|^{2} (\log n)^{1/2} \right\}.$$

Finally, we notice that

$$\begin{split} \max_{\substack{m \in \mathcal{M} \\ 0 \leq \ell, \ell' \leq p}} \left| \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m}^{2}(\boldsymbol{z}_{j}) X_{i\ell} X_{i\ell'} - \Phi_{m,\ell,\ell'} \right| \\ &= \max_{\substack{m \in \mathcal{M} \\ 0 \leq \ell, \ell' \leq p}} \left| n^{-1} \sum_{i=1}^{n} \varsigma_{i,m} - E \varsigma_{i,m} \right| + \left| E X_{i\ell} X_{i\ell'} \right| \max_{m \in \mathcal{M}} \left| \frac{1}{N} \sum_{j=1}^{N} B_{m}^{2}(\boldsymbol{z}_{j}) - \int_{\Omega} B_{m}^{2}(\boldsymbol{z}) d\boldsymbol{z} \right| \\ &= O_{a.s.} \left\{ n^{-1/2} |\Delta|^{2} (\log n)^{1/2} \right\} + O(N^{-1/2} |\Delta|). \end{split}$$

We obtain the desired result.

The following lemma provide the uniform convergence rate at which the empirical inner product in (A.1) approximates the theoretical inner product in (A.2).

Lemma A.6. Let $g_{\ell}^{(1)}(\boldsymbol{z}) = \sum_{m \in \mathcal{M}} c_{\ell m}^{(1)} B_m(\boldsymbol{z}), \ g_{\ell}^{(2)}(\boldsymbol{z}) = \sum_{m \in \mathcal{M}} c_{\ell m}^{(2)} B_m(\boldsymbol{z})$ be any spline functions in $\mathcal{S}_d^r(\Delta)$. Denote $\boldsymbol{g}(\boldsymbol{z}) = (g_0(\boldsymbol{z}), \dots, g_p(\boldsymbol{z}))^{\top}$ with $g_{\ell} \in \mathcal{S}_d^r(\Delta), \ \ell = 0, \dots, p$. Suppose Assumptions (A3) and (A5) hold, and $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, then

$$R_{n,N} = \sup_{\boldsymbol{g}^{(1)},\boldsymbol{g}^{(2)} \in \mathcal{S}_d^r(\triangle)} \left| \frac{\left\langle \boldsymbol{g}^{(1)},\boldsymbol{g}^{(2)} \right\rangle_{n,N} - \left\langle \boldsymbol{g}^{(1)},\boldsymbol{g}^{(2)} \right\rangle}{\|\boldsymbol{g}^{(1)}\| \|\boldsymbol{g}^{(2)}\|} \right| = O_P\{n^{-1/2}(\log n)^{1/2} + N^{-1/2}|\triangle|^{-1}\}.$$

Proof. It is easy to see

$$\langle \boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)} \rangle_{n,N} = \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \sum_{\ell=0}^{p} \sum_{m \in \mathcal{M}} c_{\ell m}^{(1)} X_{i\ell} B_{m}(\boldsymbol{z}_{j}) \right\} \left\{ \sum_{\ell'=0}^{p} \sum_{m' \in \mathcal{M}} c_{\ell' m'}^{(2)} X_{i\ell'} B_{m'}(\boldsymbol{z}_{j}) \right\}$$

$$= \sum_{\ell,m} \sum_{\ell',m'} c_{\ell m}^{(1)} c_{\ell' m'}^{(2)} \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} X_{i\ell} X_{i\ell'} B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j}).$$

Note that $\|\boldsymbol{g}^{(r)}\|^2 = \sum_{\ell,m} \sum_{\ell',m'} c_{\ell m}^{(r)} c_{\ell' m'}^{(r)} E(X_{\ell} X_{\ell'}) \int_{\Omega} B_m(\boldsymbol{z}) B_{m'}(\boldsymbol{z}) d\boldsymbol{z}, r = 1, 2$. It follows from Assumptions (A1), (A2), Lemmas A.1 and A.3 that,

$$c_{v}|\Delta|^{2} \sum_{\ell,m} \{c_{\ell m}^{(v)}\}^{2} \leq \|\boldsymbol{g}^{(v)}\|^{2} \leq C_{v}|\Delta|^{2} \sum_{\ell,m} \{c_{\ell m}^{(v)}\}^{2},$$

$$C_{1}|\Delta|^{2} \left[\sum_{\ell,m} \{c_{\ell m}^{(1)}\}^{2} \sum_{\ell',m'} \{c_{\ell'm'}^{(2)}\}^{2} \right]^{1/2} \leq \|\boldsymbol{g}^{(1)}\| \|\boldsymbol{g}^{(2)}\| \leq C_{2}|\Delta|^{2} \left[\sum_{\ell,m} \{c_{\ell m}^{(1)}\}^{2} \sum_{\ell',m'} \{c_{\ell'm'}^{(2)}\}^{2} \right]^{1/2}.$$

With the above preparation, we have

$$R_{n,N} \leq \frac{\sum_{\ell,\ell',|m-m'|\leq (d+2)(d+1)/2} |c_{\ell m}^{(1)} c_{\ell'm'}^{(2)}|}{C_{1}|\Delta|^{2} \left[\sum_{\ell,m} \{c_{\ell m}^{(1)}\}^{2} \sum_{\ell',m'} \{c_{\ell'm'}^{(2)}\}^{2}\right]^{1/2}}$$

$$\times \max_{\substack{m,m'\in\mathcal{M}\\0\leq\ell,\ell'\leq p}} \left|\frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j}) X_{i\ell} X_{i\ell'} - E(X_{\ell}X_{\ell'}) \int_{\Omega} B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}) d\boldsymbol{z}\right|$$

$$\leq \frac{C}{|\Delta|^{2}} \max_{\substack{m,m'\in\mathcal{M}\\0\leq\ell,\ell'\leq n}} \left|\frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m}(\boldsymbol{z}_{j}) B_{m'}(\boldsymbol{z}_{j}) X_{i\ell} X_{i\ell'} - E(X_{\ell}X_{\ell'}) \int_{\Omega} B_{m}(\boldsymbol{z}) B_{m'}(\boldsymbol{z}) d\boldsymbol{z}\right|.$$

The desired result follows from (A.7) and Lemma A.5.

As a direct result of Lemma A.6, we can see that

$$\sup_{\boldsymbol{g} \in \mathcal{S}_{r}^{r}(\triangle)} \left| \|\boldsymbol{g}\|_{n,N}^{2} / \|\boldsymbol{g}\|^{2} - 1 \right| = O_{P} \left\{ n^{-1/2} (\log n)^{1/2} + N^{-1/2} |\Delta|^{-1} \right\}.$$
 (A.8)

A.2. Uniform convergence of the unpenalized spline estimators

In this section, we consider the unpenalized spline smoothing approach. The unpenalized bivariate spline estimator of $\boldsymbol{\beta}^o = (\beta_0^o, \dots, \beta_p^o)^{\top}$ is defined as

$$\widetilde{\boldsymbol{\beta}} = (\widetilde{\beta}_0, \dots, \widetilde{\beta}_p)^{\top} = \underset{\boldsymbol{\beta} \in \mathcal{G}^{(p+1)}}{\operatorname{arg\,min}} \sum_{i=1}^n \sum_{j=1}^N \left\{ Y_i(\boldsymbol{z}_j) - \sum_{\ell=0}^p X_{i\ell} \beta_{\ell}(\boldsymbol{z}_j) \right\}^2. \tag{A.9}$$

Denote

$$egin{aligned} \widetilde{oldsymbol{ heta}}_{\mu} &= (\widetilde{oldsymbol{ heta}}_{\mu,0}^{ op}, \ldots, \widetilde{oldsymbol{ heta}}_{\mu,p}^{ op})^{ op} = \mathbf{\Gamma}_{n,0}^{-1} rac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(oldsymbol{z}_{j})
ight\} \widetilde{\mathbf{X}}_{i}^{ op} oldsymbol{ heta}^{o}(oldsymbol{z}_{j}) \ \widetilde{oldsymbol{ heta}}_{\eta} &= (\widetilde{oldsymbol{ heta}}_{\eta,0}^{ op}, \ldots, \widetilde{oldsymbol{ heta}}_{\eta,p}^{ op})^{ op} = \mathbf{\Gamma}_{n,0}^{-1} rac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(oldsymbol{z}_{j})
ight\} \eta_{i}(oldsymbol{z}_{j}), \ \widetilde{oldsymbol{ heta}}_{arepsilon} &= (\widetilde{oldsymbol{ heta}}_{arepsilon,0}^{ op}, \ldots, \widetilde{oldsymbol{ heta}}_{arepsilon,p}^{ op})^{ op} = \mathbf{\Gamma}_{n,0}^{-1} rac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(oldsymbol{z}_{j})
ight\} \sigma(oldsymbol{z}_{j}) arepsilon_{ij}, \end{aligned}$$

where

$$\Gamma_{n,0} = \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} (\widetilde{\mathbf{X}}_{i} \widetilde{\mathbf{X}}_{i}^{\top}) \otimes \{\widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}^{\top}(\boldsymbol{z}_{j})\}.$$
(A.10)

Lemma A.7. Under Assumptions (A3) and (A5), if $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, then there exist constants $0 < c_{\Gamma} < C_{\Gamma} < \infty$, such that with probability approaching 1, as $N \to \infty$, $n \to \infty$, $c_{\Gamma}|\Delta|^2 \le \lambda_{\min}(\Gamma_{n,0}) \le \lambda_{\max}(\Gamma_{n,0}) \le C_{\Gamma}|\Delta|^2$, where $\Gamma_{n,0}$ is in (A.10).

Proof. Note that for any vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_0^\top, \cdots, \boldsymbol{\theta}_p^\top)^\top$ with $\boldsymbol{\gamma}_\ell = (\gamma_{\ell m}, m \in \mathcal{M})^\top$

$$\boldsymbol{\theta}^{\top} \boldsymbol{\Gamma}_{n,0} \boldsymbol{\theta} = \frac{1}{nN} \boldsymbol{\gamma}^{\top} \sum_{i=1}^{n} \sum_{j=1}^{N} (\widetilde{\mathbf{X}}_{i} \widetilde{\mathbf{X}}_{i}^{\top}) \otimes \{\mathbf{B}(\boldsymbol{z}_{j}) \mathbf{B}^{\top}(\boldsymbol{z}_{j})\} \boldsymbol{\gamma} = \|\boldsymbol{g}_{\boldsymbol{\gamma}}\|_{n,N}^{2}, \tag{A.11}$$

where $\gamma = \mathbf{Q}_2 \boldsymbol{\theta}$, and $\boldsymbol{g}_{\gamma} = (g_{\gamma_0}, \dots, g_{\gamma_p})^{\top}$ with $g_{\gamma_{\ell}} = \sum_{m \in \mathcal{M}} \gamma_{\ell m} B_m$. By (A.8), we have $c(1 - R_{n,N}) |\Delta|^2 ||\gamma||^2 \le (1 - R_{n,N}) ||\boldsymbol{g}_{\gamma}||^2 \le (1 + R_{n,N}) ||\boldsymbol{g}_{\gamma}||^2 \le C(1 + R_{n,N}) |\Delta|^2 ||\gamma||^2$,

in which we have used the stability conditions in Lemma A.1.

Next, we consider the following decomposition $\widetilde{\boldsymbol{\beta}}(\boldsymbol{z}) = \widetilde{\boldsymbol{\beta}}_{\mu}(\boldsymbol{z}) + \widetilde{\boldsymbol{\gamma}}(\boldsymbol{z}) + \widetilde{\boldsymbol{\varepsilon}}(\boldsymbol{z})$, where

$$\widetilde{\boldsymbol{\beta}}_{\mu}(\boldsymbol{z}) = (\widetilde{\beta}_{\mu,0}(\boldsymbol{z}), \dots, \widetilde{\beta}_{\mu,p}(\boldsymbol{z}))^{\top} = \{\mathbf{I} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} \widetilde{\boldsymbol{\theta}}_{\mu},$$
 (A.12)

$$\widetilde{\boldsymbol{\eta}}(\boldsymbol{z}) = (\widetilde{\eta}_0(\boldsymbol{z}), \dots, \widetilde{\eta}_p(\boldsymbol{z}))^{\top} = \{\mathbf{I} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} \widetilde{\boldsymbol{\theta}}_n,$$
 (A.13)

$$\widetilde{\boldsymbol{\varepsilon}}(\boldsymbol{z}) = (\widetilde{\varepsilon}_0(\boldsymbol{z}), \dots, \widetilde{\varepsilon}_p(\boldsymbol{z}))^{\top} = \{\mathbf{I} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} \widetilde{\boldsymbol{\theta}}_{\varepsilon}.$$
 (A.14)

Lemma A.8. Under Assumptions (A2)-(A5) and (C1), if $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, $\|\sum_{k=1}^{\infty} \lambda_k^{1/2} \psi_k\|_{\infty} < \infty$ and $n^{1/(4+\delta_2)} \ll n^{1/2} N^{-1/2} |\Delta|^{-1}$ for some δ_2 , then for $\widetilde{\boldsymbol{\eta}}$ and $\widetilde{\boldsymbol{\varepsilon}}$ in (A.13) and (A.14), $\|\widetilde{\boldsymbol{\eta}}\|_{\infty} = O_P\{n^{-1/2}(\log n)^{1/2}\}$ and $\|\widetilde{\boldsymbol{\varepsilon}}\|_{\infty} = O_P\{(nN)^{-1/2}(\log n)^{1/2}|\Delta|^{-1}\}$.

Proof. Note that for any $\ell = 0, 1, ..., p$, $\widetilde{\eta}_{\ell}(\boldsymbol{z}) = \sum_{m \in \mathcal{M}} \widetilde{\theta}_{\eta, \ell, m} \widetilde{B}_{m}(\boldsymbol{z})$ for some coefficients $\widetilde{\theta}_{\eta, \ell, m}$, so the order of $\widetilde{\eta}_{\ell}(\boldsymbol{z})$ is related to that of $\widetilde{\theta}_{\eta, \ell, m}$. In fact

$$\|\widetilde{\boldsymbol{\eta}}\|_{\infty} = \max_{0 \le \ell \le p} \|\widetilde{\eta}_{\ell}\|_{\infty} \le C_{\eta} \|\widetilde{\boldsymbol{\theta}}_{\eta,\ell}\|_{\infty} = \left\| (\mathbf{e}_{\ell} \otimes \mathbf{1})^{\top} \mathbf{\Gamma}_{n,0}^{-1} \left[\frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \eta_{i}(\boldsymbol{z}_{j}) \right] \right\|_{\infty},$$

where $\widetilde{\boldsymbol{\theta}}_{\eta} = (\widetilde{\boldsymbol{\theta}}_{\eta,\ell,m})_{m \in \widetilde{\mathcal{M}}}$ with $\widetilde{\mathcal{M}}$ being an index set of the transformed Bernstein basis polynomials $\widetilde{B}_m(\boldsymbol{z})$ and $\Gamma_{n,0}$ is the symmetric positive definite matrix defined in (A.10). Thus, by Lemma A.7,

$$\|\widetilde{\boldsymbol{\eta}}\|_{\infty} \leq C|\Delta|^{-2} \max_{0 \leq \ell \leq p} \max_{m \in \widetilde{\mathcal{M}}} \left| \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} X_{i\ell} \eta_i(\boldsymbol{z}_j) \widetilde{B}_m(\boldsymbol{z}_j) \right|,$$

almost surely. Next, we show that with probability 1,

$$\max_{0 \le \ell \le p} \max_{m \in \widetilde{\mathcal{M}}} \left| \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} X_{i\ell} \eta_i(\boldsymbol{z}_j) \widetilde{B}_m(\boldsymbol{z}_j) \right| = O\left\{ n^{-1/2} |\Delta|^2 (\log n)^{1/2} \right\}. \tag{A.15}$$

To prove (A.15), let $\varpi_i = \varpi_{i,m} = \sum_{k=1}^{\infty} \lambda_k^{1/2} X_{i\ell} \xi_{ik} \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\boldsymbol{z}_j) \psi_k(\boldsymbol{z}_j)$, where $E(\varpi_i) = 0$ and

$$E(\varpi_i^2) = \frac{E(X_{i\ell}^2)}{N^2} \sum_{j=1}^N \sum_{j'=1}^N \widetilde{B}_m(\boldsymbol{z}_j) \widetilde{B}_m(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'})$$

$$\approx \int_{\Omega^2} G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_m(\boldsymbol{z}) B_m(\boldsymbol{z}') d\boldsymbol{z} d\boldsymbol{z}' \approx |\triangle|^4.$$

We decompose the random variable ϖ_i into a tail part and a truncated part,

$$\varpi_{i,1}^{D_n} = \sum_{k=1}^{\infty} \lambda_k^{1/2} \left\{ \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\mathbf{z}_j) \psi_k(\mathbf{z}_j) \right\} X_{i\ell} \xi_{ik} I \left\{ |X_{i\ell} \xi_{ik}| > D_n \right\},
\varpi_{i,2}^{D_n} = \sum_{k=1}^{\infty} \lambda_k^{1/2} \left\{ \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\mathbf{z}_j) \psi_k(\mathbf{z}_j) \right\} X_{i\ell} \xi_{ik} I \left\{ |X_{i\ell} \xi_{ik}| \le D_n \right\} - \mu_i^{D_n},
\mu_i^{D_n} = \sum_{k=1}^{\infty} \lambda_k^{1/2} \left\{ \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\mathbf{z}_j) \psi_k(\mathbf{z}_j) \right\} E \left[X_{i\ell} \xi_{ik} I \left\{ |X_{i\ell} \xi_{ik}| \le D_n \right\} \right],$$

where $D_n = n^{\alpha} (1/(4 + \delta_1) < \alpha < 1/2)$. At first, we show that tail part vanishes almost surely. Note that, for any $k \ge 1$,

$$\sum_{n=1}^{\infty} P\{|X_{n\ell}\xi_{nk}| > D_n\} \le \sum_{n=1}^{\infty} \frac{E|X_{n\ell}\xi_{nk}|^{4+\delta_1}}{D_n^{4+\delta_1}} \le \upsilon_{\delta_1} \sum_{n=1}^{\infty} D_n^{-(4+\delta_1)} < \infty.$$

By the Borel-Cantelli's lemma, we can show that $E\left|\frac{1}{n}\sum_{i=1}^{n}\varpi_{i,1}^{D_{n}}\right|=O(n^{-r})$, for any r>0. As $E(\varpi_{i})=0$, then it is straightforward to verify that $\mu_{i}^{D_{n}}=-E(\varpi_{i,1}^{D_{n}})=O(D_{n}^{-2}|\Delta|^{2})$.

Next, notice that $E(\varpi_{i,2}^{D_n})=0$. Then, $\operatorname{Var}(\varpi_{i,2}^{D_n})=E(\varpi_i^2)-E(\varpi_{i,1}^{D_n})^2-(\mu^{D_n})^2 \asymp |\Delta|^4$. Also, we have, for any $r\geq 3$,

$$E|\varpi_{i,2}^{D_n}|^r = E\left|\sum_{k=1}^{\infty} \lambda_k^{1/2} \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\boldsymbol{z}_j) \psi_k(\boldsymbol{z}_j) \left[X_{i\ell} \xi_{ik} I \left\{ |X_{i\ell} \xi_{ik}| \leq D_n \right\} \right] - \mu_i^{D_n} \right|^r$$

$$\leq 2^{r-1} \left[E\left|\sum_{k=1}^{\infty} \lambda_r^{1/2} \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\boldsymbol{z}_j) \psi_k(\boldsymbol{z}_j) X_{i\ell} \xi_{ik} I \left\{ |X_{i\ell} \xi_{ik}| \leq D_n \right\} \right|^r + \left(\mu_i^{D_n}\right)^r \right]$$

$$\leq \left\{ 2D_n \frac{1}{N} \sum_{j=1}^{N} \widetilde{B}_m(\boldsymbol{z}_j) \sum_{k=1}^{\infty} \lambda_k^{1/2} \psi_k(\boldsymbol{z}_j) \right\}^{r-2} E|\varpi_{i,2}^{D_n}|^2 \leq (CD_n|\Delta|^2)^{r-2} E|\varpi_{i,2}^{D_n}|^2.$$

Thus, $E |\varpi_{i,2}/n|^r \leq \{Cn^{-1}D_n|\Delta|^2\}^{r-2}r!E(\varpi_{i,2}^2/n^2) < \infty$ with the Cramer constant $c^* = Cn^{-1}D_n|\Delta|^2$. By the Bernstein inequality, for any large enough $\delta > 0$,

$$P\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \varpi_i^{D_n} \right| \ge \delta n^{-1/2} |\Delta|^2 (\log n)^{1/2} \right\} \le 2 \exp\left\{ \frac{-\delta^2 \log n}{4c + 2\delta C D_n (\log n)^{1/2} n^{-1/2}} \right\} \le 2n^{-3}.$$

Hence,

$$\sum_{n=1}^{\infty} P\left\{\max_{0\leq \ell\leq p}\max_{m\in\mathcal{M}}\left|\frac{1}{n}\sum_{i=1}^{n}\varpi_{i}\right|\geq \delta n^{-1/2}|\triangle|^{2}(\log n)^{1/2}\right\}\leq C|\triangle|^{-2}\sum_{n=1}^{\infty}n^{-3}<\infty$$

for such $\delta > 0$. Thus, Borel-Cantelli's lemma implies that $\|\widetilde{\boldsymbol{\eta}}\|_{\infty} = O_P\{n^{-1/2}(\log n)^{1/2}\}$. The result of $\|\widetilde{\boldsymbol{\varepsilon}}\|_{\infty} = O_P\{(nN)^{-1/2}(\log n)^{1/2}|\Delta|^{-1}\}$ can be established similarly, thus omitted.

For $\widetilde{\boldsymbol{\beta}}(\boldsymbol{z})$ defined in (A.9), Theorem A.5 below provides its uniform convergence rate to $\boldsymbol{\beta}^o$.

Theorem A.5. Under Assumptions (A1)-(A6), for $\widetilde{\boldsymbol{\beta}}(\boldsymbol{z})$ defined in (A.9), $\|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^o\|_{\infty} = O_P\{|\Delta|^{d+1}\|\boldsymbol{\beta}^o\|_{d+1,\infty} + n^{-1/2}(\log n)^{1/2}\}.$

Proof. Note that $\|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^o\|_{\infty} \leq \|\widetilde{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^o\|_{\infty} + \|\widetilde{\boldsymbol{\eta}}\|_{\infty} + \|\widetilde{\boldsymbol{\varepsilon}}\|_{\infty}$, where

$$\widetilde{\boldsymbol{\beta}}_{\mu} = \operatorname*{arg\,min}_{\boldsymbol{g} \in \mathcal{G}^{(p+1)}} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \sum_{\ell=0}^{p} X_{i\ell} (\beta_{\ell}^{o} - g_{\ell})(\boldsymbol{z}_{j}) \right\}^{2}.$$

Let $\boldsymbol{\beta}^* = (\beta_0^*, \dots, \beta_p^*)^* \in \mathcal{G}^{(p+1)}$, where β_ℓ^* 's are the best approximation to β_ℓ^o 's with the approximation rate $\|\beta_\ell^* - \beta_\ell^o\|_{\infty} \leq C|\Delta|^{d+1}\|\boldsymbol{\beta}^o\|_{d+1,\infty}$ for any $\ell = 0, \dots, p$. By Lai and Wang (2013),

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^{o}\|_{\infty} \leq \|\widetilde{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^{*}\|_{\infty} + \|\boldsymbol{\beta}^{*} - \boldsymbol{\beta}^{o}\|_{\infty} \leq C|\Delta|^{d+1}\|\boldsymbol{\beta}^{o}\|_{d+1,\infty}. \tag{A.16}$$

The desired result follows from Lemma A.8.

A.3. Asymptotic properties of penalized spline estimators

Let $\widetilde{\mathbf{B}}(z) = \mathbf{Q}_2^{\mathsf{T}} \mathbf{B}(z)$, then for $\mathbb{U} = \mathbf{X} \otimes (\mathbf{B} \mathbf{Q}_2)$ defined in Section 2.2, we have

$$\mathbb{U}^{ op} = (\widetilde{\mathbf{X}}_1 \otimes \widetilde{\mathbf{B}}({oldsymbol{z}}_1), \ldots, \widetilde{\mathbf{X}}_1 \otimes \widetilde{\mathbf{B}}({oldsymbol{z}}_N), \ldots, \widetilde{\mathbf{X}}_n \otimes \widetilde{\mathbf{B}}({oldsymbol{z}}_1), \ldots, \widetilde{\mathbf{X}}_n \otimes \widetilde{\mathbf{B}}({oldsymbol{z}}_N)),$$

and $\mathbb{U}^{\top}\mathbb{U} = \sum_{i=1}^{n} \sum_{j=1}^{N} (\widetilde{\mathbf{X}}_{i}\widetilde{\mathbf{X}}_{i}^{\top}) \otimes \{\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}^{\top}(\boldsymbol{z}_{j})\}, \ \mathbb{U}^{\top}\mathbb{Y} = \sum_{i=1}^{n} \sum_{j=1}^{N} \{\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\}Y_{ij}.$ Let

$$\mathbf{\Gamma}_{n,\rho} = \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} (\widetilde{\mathbf{X}}_{i} \widetilde{\mathbf{X}}_{i}^{\top}) \otimes \{\widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}^{\top}(\boldsymbol{z}_{j})\} + \frac{\rho_{n}}{nN} \mathbf{I}_{p} \otimes \mathbf{Q}_{2}^{\top} [\langle B_{m}, B_{m'} \rangle_{\mathcal{E}}]_{m,m' \in \mathcal{M}} \mathbf{Q}_{2},$$
(A.17)

which is a symmetric positive definite matrix.

Next, we define

$$\begin{split} \widehat{\boldsymbol{\theta}}_{\mu} &= (\widehat{\boldsymbol{\theta}}_{\mu,0}^{\top}, \dots, \widehat{\boldsymbol{\theta}}_{\mu,p}^{\top})^{\top} = \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \widetilde{\mathbf{X}}_{i}^{\top} \boldsymbol{\beta}^{o}(\boldsymbol{z}_{j}), \\ \widehat{\boldsymbol{\theta}}_{\eta} &= (\widehat{\boldsymbol{\theta}}_{\eta,0}^{\top}, \dots, \widehat{\boldsymbol{\theta}}_{\eta,p}^{\top})^{\top} = \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}), \\ \widehat{\boldsymbol{\theta}}_{\varepsilon} &= (\widehat{\boldsymbol{\theta}}_{\varepsilon,0}^{\top}, \dots, \widehat{\boldsymbol{\theta}}_{\varepsilon,p}^{\top})^{\top} = \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij}. \end{split}$$

Note that, for any $\ell = 0, \ldots, p$, the penalized bivariate spline estimator $\widehat{\beta}_{\ell}$ can be written as:

$$\widehat{\beta}_{\ell}(z) = \widehat{\beta}_{\mu,\ell}(z) + \widehat{\eta}_{\ell}(z) + \widehat{\varepsilon}_{\ell}(z), \tag{A.18}$$

where

$$\widehat{eta}_{\mu,\ell}(oldsymbol{z}) = \widetilde{\mathbf{B}}(oldsymbol{z})^ op \widehat{oldsymbol{ heta}}_{\mu,\ell}, \ \ \widehat{\eta}_\ell(oldsymbol{z}) = \widetilde{\mathbf{B}}(oldsymbol{z})^ op \widehat{oldsymbol{ heta}}_{\eta,\ell}, \ \ \widehat{arepsilon}_\ell(oldsymbol{z}) = \widetilde{\mathbf{B}}(oldsymbol{z})^ op \widehat{oldsymbol{ heta}}_{arepsilon,\ell},$$

Therefore, we have

$$\widehat{\beta}_{\ell}(z) - \beta_{\ell}^{o}(z) = \widehat{\beta}_{\mu,\ell}(z) - \beta_{\ell}^{o}(z) + \widehat{\eta}_{\ell}(z) + \widehat{\varepsilon}_{\ell}(z). \tag{A.19}$$

Lemma A.9. Under Assumptions (A3)-(A5), if $N^{1/2}|\triangle| \to \infty$ as $N \to \infty$, then there exist constants $0 < c_{\Gamma} < C_{\Gamma} < \infty$, such that with probability approaching 1 as $N \to \infty$ and $n \to \infty$, $c_{\Gamma}|\triangle|^2 \le \lambda_{\min}(\Gamma_{n,\rho}) \le \lambda_{\max}(\Gamma_{n,\rho}) \le C_{\Gamma}\left(|\triangle|^2 + \frac{\rho_n}{nN|\triangle|^2}\right)$.

Proof. By (A.11), it is easy to see that, for any vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_0^\top, \cdots, \boldsymbol{\theta}_p^\top)^\top$,

$$oldsymbol{ heta}^{ op} oldsymbol{\Gamma}_{n,
ho} oldsymbol{ heta} = \|oldsymbol{g}_{oldsymbol{\gamma}}\|_{n,N}^2 + rac{
ho_n}{nN} \sum_{\ell=0}^p oldsymbol{\gamma}_{\ell}^{ op} [\langle B_m, B_{m'}
angle_{\mathcal{E}}]_{m,m' \in \mathcal{M}} oldsymbol{\gamma}_{\ell},$$

where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_p)^{\top} = \mathbf{Q}_2 \boldsymbol{\theta}$ with $\boldsymbol{\gamma}_{\ell} = (\gamma_{\ell m}, m \in \mathcal{M})^{\top}$. Using the Markov's inequality in the supplement of Lai and Wang (2013) and Lemma A.1, we have

$$\sum_{\ell=0}^{p} \left\| \sum_{m \in \mathcal{M}} \gamma_{\ell m} B_m \right\|_{\mathcal{E}}^{2} \leq \frac{C}{|\Delta|^4} \sum_{\ell=0}^{p} \left\| \sum_{m \in \mathcal{M}} \gamma_{\ell m} B_m \right\|_{L_2}^{2} \leq \frac{C}{|\Delta|^2} \|\boldsymbol{\gamma}\|^{2}.$$

Thus, the largest eigenvalue of the matrix $\Gamma_{n,\rho}$ in (A.17) satisfies that $\lambda_{\max}(\Gamma_{n,\rho}) \leq C\{(1+R_{n,N})|\Delta|^2+(nN|\Delta|^2)^{-1}\rho_n\}$. Thus, we have with probability approaching 1, $\lambda_{\max}(\Gamma_{n,\rho}) \leq C_{\Gamma}\{|\Delta|^2+(nN|\Delta|^2)^{-1}\rho_n\}$ for some positive constant C_{Γ} . On the other hand, we use Lemma A.1 and equation (A.8) to have $\|\boldsymbol{g}_{\gamma}\|_{n,N}^2 = (1-R_{n,N})\|\boldsymbol{g}_{\gamma}\|^2 \geq c(1-R_{n,N})|\Delta|^2\|\boldsymbol{\gamma}\|^2$.

Therefore,
$$\lambda_{\min}(\Gamma_{n,\rho}) \geq c(1 - R_{n,N})|\Delta|^2 = c_{\Gamma}|\Delta|^2$$
.

Lemma A.10. Under Assumptions (A1), (A3) and (A5), if $N^{1/2}|\triangle| \to \infty$, one has $\|\widehat{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^o\|_{\infty} = O_P\left\{\frac{\rho_n}{nN|\triangle|^3}\|\boldsymbol{\beta}^o\|_{2,\infty} + \left(1 + \frac{\rho_n}{nN|\triangle|^5}\right)|\triangle|^{d+1}\|\boldsymbol{\beta}^o\|_{d+1,\infty}\right\}$.

Proof. Define

$$A_n = \sup_{\boldsymbol{g} \in \mathcal{G}^{(p+1)}} \left\{ \frac{\|\boldsymbol{g}\|_{\infty}}{\|\boldsymbol{g}\|_{n,N}}, \|\boldsymbol{g}\|_{n,N} \neq 0 \right\}, \quad \overline{A}_n = \sup_{\boldsymbol{g} \in \mathcal{G}^{(p+1)}} \left\{ \frac{\|\boldsymbol{g}\|_{\mathcal{E}}}{\|\boldsymbol{g}\|_{n,N}}, \|\boldsymbol{g}\|_{n,N} \neq 0 \right\}, \quad (A.20)$$

where random variables A_n and \overline{A}_n depend on the collection of $X_{i\ell}$'s, $i = 1, \ldots, n$, $\ell = 0, \ldots, p$. It is clear that $\|\boldsymbol{\beta}^o - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty} \leq \|\boldsymbol{\beta}^o - \widetilde{\boldsymbol{\beta}}_{\mu}\|_{\infty} + \|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty}$, where $\widetilde{\boldsymbol{\beta}}_{\mu}$ is given in (A.12), and $\|\widetilde{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^o\|_{\infty} \leq C|\Delta|^{d+1}\|\boldsymbol{\beta}^o\|_{d+1,\infty}$ according to (A.16).

By the definition of A_n in (A.20), we have

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty} \le A_n \|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N}. \tag{A.21}$$

Note that the penalized spline $\widehat{\beta}_{\mu}$ of β^{o} is characterized by the orthogonality relations

$$nN\langle \boldsymbol{\beta}^o - \widehat{\boldsymbol{\beta}}_{\mu}, \boldsymbol{g} \rangle_{n,N} = \rho_n \langle \widehat{\boldsymbol{\beta}}_{\mu}, \boldsymbol{g} \rangle_{\mathcal{E}}, \quad \text{for all } \boldsymbol{g} \in \mathcal{G}^{(p+1)},$$
 (A.22)

while $\widetilde{\boldsymbol{\beta}}_{u}$ is characterized by

$$\langle \boldsymbol{\beta}^o - \widetilde{\boldsymbol{\beta}}_{\mu}, \boldsymbol{g} \rangle_{n,N} = 0, \text{ for all } \boldsymbol{g} \in \mathcal{G}^{(p+1)}.$$
 (A.23)

By (A.22) and (A.23), we have $nN\langle \widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}, \boldsymbol{g} \rangle_{n,N} = \rho_n \langle \widehat{\boldsymbol{\beta}}_{\mu}, \boldsymbol{g} \rangle_{\mathcal{E}}$, for all $\boldsymbol{g} \in \mathcal{G}^{(p+1)}$. Inserting $\boldsymbol{g} = \widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}$ yields that

$$nN\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N}^{2} = \rho_{n}\langle\widehat{\boldsymbol{\beta}}_{\mu}, \widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\rangle_{\mathcal{E}}.$$
 (A.24)

Thus, by Cauchy-Schwarz inequality and the definition of \overline{A}_n .

$$nN\|\widetilde{\boldsymbol{\beta}}_{\mu}-\widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N}^{2} \leq \rho_{n}\|\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}\|\widetilde{\boldsymbol{\beta}}_{\mu}-\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}} \leq \rho_{n}\overline{A}_{n}\|\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}\|\widetilde{\boldsymbol{\beta}}_{\mu}-\widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N}.$$

Similarly, using (A.24), $nN\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N}^2 = \rho_n\{\langle\widehat{\boldsymbol{\beta}}_{\mu},\widetilde{\boldsymbol{\beta}}_{\mu}\rangle_{\mathcal{E}} - \langle\widehat{\boldsymbol{\beta}}_{\mu},\widehat{\boldsymbol{\beta}}_{\mu}\rangle_{\mathcal{E}}\} \geq 0$. Thus, by Cauchy-Schwarz inequality, $\|\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}^2 \leq \langle\widehat{\boldsymbol{\beta}}_{\mu},\widetilde{\boldsymbol{\beta}}_{\mu}\rangle_{\mathcal{E}} \leq \|\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}\|\widetilde{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}$, which implies that $\|\widehat{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}} \leq \|\widetilde{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}$. Therefore,

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N} \le \rho_n (nN)^{-1} \overline{A}_n \|\widetilde{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}. \tag{A.25}$$

Combining (A.21) and (A.25) yields that

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty} \leq A_{n} \|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{n,N} \leq \rho_{n}(nN)^{-1} A_{n} \overline{A}_{n} \|\widetilde{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}}.$$

By Lemma A.2, we have

$$\|\widetilde{\boldsymbol{\beta}}_{\mu}\|_{\mathcal{E}} = C_1\{\|\boldsymbol{\beta}^o\|_{2,\infty} + \sum_{a_1+a_2=2} \|\nabla_{z_1}^{a_1}\nabla_{z_2}^{a_2}(\boldsymbol{\beta}^o - \widetilde{\boldsymbol{\beta}}_{\mu})\|_{\infty}\} \leq C_2(\|\boldsymbol{\beta}^o\|_{2,\infty} + |\Delta|^{d-1}\|\boldsymbol{\beta}^o\|_{d+1,\infty}).$$

It follows

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty} = \rho_n(nN)^{-1} A_n \overline{A}_n C_2(\|\boldsymbol{\beta}^o\|_{2,\infty} + |\Delta|^{d-1} \|\boldsymbol{\beta}^o\|_{d+1,\infty}). \tag{A.26}$$

Next we derive the order of A_n and \overline{A}_n . By Markov's inequality, for any $\mathbf{g} \in \mathcal{G}^{(p+1)}$, $\|\mathbf{g}\|_{\infty} \leq C|\Delta|^{-1}\|\mathbf{g}\|$, $\|\mathbf{g}\|_{\mathcal{E}} \leq C|\Delta|^{-2}\|\mathbf{g}\|$. Equation (A.8) implies that

$$\sup_{\boldsymbol{g} \in \mathcal{G}(\triangle)} \left\{ \|\boldsymbol{g}\|_{n,N} / \|\boldsymbol{g}\| \right\} \ge \left[1 - O_P \left\{ (\log n)^{1/2} n^{-1/2} + N^{-1/2} |\triangle|^{-1} \right\} \right]^{1/2}.$$

Thus, we have

$$A_n \le C|\Delta|^{-1} \left[1 - O_P \left\{ (\log n)^{1/2} n^{-1/2} + N^{-1/2} |\Delta|^{-1} \right\} \right]^{-1/2} = O_P \left(|\Delta|^{-1} \right),$$

$$\overline{A}_n \le C|\Delta|^{-2} \left[1 - O_P \left\{ (\log n)^{1/2} n^{-1/2} + N^{-1/2} |\Delta|^{-1} \right\} \right]^{-1/2} = O_P \left(|\Delta|^{-2} \right).$$

Plugging the order of A_n and \overline{A}_n into (A.26) yields that

$$\|\widetilde{\boldsymbol{\beta}}_{\mu} - \widehat{\boldsymbol{\beta}}_{\mu}\|_{\infty} = O_{P} \left\{ \frac{C_{2} \rho_{n}}{n N |\Delta|^{3}} (\|\boldsymbol{\beta}^{o}\|_{2,\infty} + |\Delta|^{d-1} \|\boldsymbol{\beta}^{o}\|_{d+1,\infty}) \right\}.$$

Hence,

$$\|\widehat{\boldsymbol{\beta}}_{\mu} - \boldsymbol{\beta}^{o}\|_{\infty} \leq C_{1}|\Delta|^{d+1}\|\boldsymbol{\beta}^{o}\|_{d+1,\infty} + O_{P}\left\{\frac{C_{2}\rho_{n}}{nN|\Delta|^{3}}\left(\|\boldsymbol{\beta}^{o}\|_{2,\infty} + |\Delta|^{d-1}\|\boldsymbol{\beta}^{o}\|_{d+1,\infty}\right)\right\}.$$

Therefore, Lemma A.10 is established.

Lemma A.11. Suppose Assumptions (A2)-(A5) hold and $N^{1/2}|\triangle| \to \infty$ as $N \to \infty$, then $\|\widehat{\boldsymbol{\theta}}_n\|^2 = O_P(n^{-1}|\triangle|^{-2})$.

Proof. Note that $\widehat{\boldsymbol{\theta}}_{\eta} = \Gamma_{n,\rho}^{-1} \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j})$. According to Lemma A.9,

$$\begin{split} \|\widehat{\boldsymbol{\theta}}_{\eta}\|^{2} & \approx \frac{1}{n^{2}N^{2}|\triangle|^{4}} \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\}^{\top} \\ & \times \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \left\{ \mathbf{X}_{i'} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \right\} \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{i'k} \psi_{k}(\boldsymbol{z}_{j'}). \end{split}$$

Note that

$$\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j})$$

$$= \left(X_{i0} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}), \dots, X_{ip} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \sum_{k=1}^{\infty} \lambda_{k}^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \right)^{\top},$$

so one has

$$\|\widehat{\boldsymbol{\theta}}_{\eta}\|^{2} \asymp \frac{1}{n^{2}N^{2}|\Delta|^{4}} \sum_{\ell=0}^{p} \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{N} X_{i\ell} X_{i'\ell} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \sum_{k,k'=1}^{\infty} (\lambda_{k} \lambda_{k'})^{1/2} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \xi_{i'k'} \psi_{k'}(\boldsymbol{z}_{j'}).$$

Because the eigenvalues of $\mathbf{Q}_2\mathbf{Q}_2^{\mathsf{T}}$ are either 0 or 1, under Assumptions (A2) and (A3), for any ℓ, i ,

$$\frac{1}{N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} E \left\{ X_{i\ell}^2 \widetilde{\mathbf{B}}(\boldsymbol{z}_j)^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \sum_{k,k'=1}^{\infty} (\lambda_k \lambda_{k'})^{1/2} \xi_{ik} \psi_k(\boldsymbol{z}_j) \xi_{ik'} \psi_{k'}(\boldsymbol{z}_{j'}) \right\} \\
\leq C \sum_{m \in \mathcal{M}} \frac{1}{N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} B_m(\boldsymbol{z}_j) B_m(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}).$$

Assumption (A4) and (A.5) imply that

$$\frac{1}{N^2} \sum_{j \neq j'} B_m(\boldsymbol{z}_j) B_m(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) = \int_{T_m \times T_m} G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') B_m(\boldsymbol{z}) B_m(\boldsymbol{z}') d\boldsymbol{z} d\boldsymbol{z}' \\
\times \left\{ 1 + O(N^{-1/2} |\Delta|^3) \right\} = O(|\Delta|^4).$$

Thus,

$$\frac{1}{N^2} \sum_{j=1}^N \sum_{j'=1}^N E X_{i\ell}^2 \mathbf{B}(\boldsymbol{z}_j)^\top \mathbf{B}(\boldsymbol{z}_{j'}) \sum_{k,k'=1}^\infty (\lambda_k \lambda_{k'})^{1/2} \xi_{ik} \psi_k(\boldsymbol{z}_j) \xi_{ik'} \psi_{k'}(\boldsymbol{z}_{j'}) \leq C |\Delta|^2.$$

Next for any ℓ , $i \neq i'$, j, j', we have

$$E\left\{X_{i\ell}X_{i'\ell}\mathbf{B}(\boldsymbol{z}_{j})^{\top}\mathbf{B}(\boldsymbol{z}_{j'})\sum_{k,k'=1}^{\infty}(\lambda_{k}\lambda_{k'})^{1/2}\xi_{ik}\psi_{k}(\boldsymbol{z}_{j})\xi_{ik'}\psi_{k'}(\boldsymbol{z}_{j'})\right\}$$

$$=E(X_{i\ell}X_{i'\ell})\sum_{m\in\mathcal{M}}B_{m}^{2}(\boldsymbol{z}_{j})B_{m}^{2}(\boldsymbol{z}_{j'})\sum_{k,k'}E\left\{(\lambda_{k}\lambda_{k'})^{1/2}\xi_{ik}\xi_{i'k'}\psi_{k}(\boldsymbol{z}_{j})\psi_{k'}(\boldsymbol{z}_{j'})\right\}=0.$$

Therefore, $E\|\widehat{\boldsymbol{\theta}}_{\eta}\|^2 \leq Cp(n^{-1}|\Delta|^{-2})$. The conclusion of the lemma follows.

Lemma A.12. Suppose Assumptions (A2)-(A5) hold and $N^{1/2}|\triangle| \to \infty$ as $N \to \infty$, then $\|\widehat{\boldsymbol{\theta}}_{\varepsilon}\|^2 = O_P(n^{-1}N^{-1}|\triangle|^{-4})$.

Proof. By the definition of $\widehat{\boldsymbol{\theta}}_{\varepsilon}$ in (A.37), we have

$$\begin{split} \|\widehat{\boldsymbol{\theta}}_{\varepsilon}\|^{2} &= \frac{1}{n^{2}N^{2}|\triangle|^{4}} \left(|\triangle|^{-2}\boldsymbol{\Gamma}_{n,\rho}\right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\right\}^{\top} \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij} \\ &\times \left(|\triangle|^{-2}\boldsymbol{\Gamma}_{n,\rho}\right)^{-1} \sum_{i=1}^{n} \sum_{j'=1}^{N} \left\{\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right\} \sigma(\boldsymbol{z}_{j'}) \varepsilon_{ij'}. \end{split}$$

By Lemma A.9,

$$\|\widehat{m{ heta}}_{arepsilon}\|^2symp rac{1}{n^2N^2|\Delta|^4}\sum_{i,i'=1}^n\sum_{j,j'=1}^N\left\{\widetilde{\mathbf{X}}_i\otimes\widetilde{\mathbf{B}}(m{z}_j)
ight\}^ op\sigma(m{z}_j)arepsilon_{ij}\left\{\mathbf{X}_{i'}\otimes\widetilde{\mathbf{B}}(m{z}_{j'})
ight\}\sigma(m{z}_{j'})arepsilon_{i'j'}.$$

Note that

$$\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij} = \left(X_{i0} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij}, X_{i1} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij}, \dots, X_{ip} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij} \right)^{\top},$$

so one has

$$\|\widetilde{\boldsymbol{\theta}}_{\varepsilon}\|^{2} \asymp \frac{1}{n^{2}N^{2}|\triangle|^{4}} \sum_{\ell=0}^{p} \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{N} X_{i\ell} X_{i'\ell} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \sigma(\boldsymbol{z}_{j}) \sigma(\boldsymbol{z}_{j'}) \varepsilon_{ij} \varepsilon_{i'j}.$$

Because the eigenvalues of $\mathbf{Q}_2\mathbf{Q}_2^{\top}$ are either 0 or 1, under Assumption (A2), for any ℓ, i , by (A.6),

$$\frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) = \mathbf{B}(\boldsymbol{z}_{j})^{\top} \mathbf{Q}_{2} \mathbf{Q}_{2}^{\top} \mathbf{B}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) \leq C \sum_{m \in \mathcal{M}} \frac{1}{N} \sum_{j=1}^{N} B_{m}^{2}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) \\
\leq C \sum_{m \in \mathcal{M}} \int_{T_{\lceil m/d^{*} \rceil}} \sigma^{2}(\boldsymbol{z}) B_{m}^{2}(\boldsymbol{z}) d\boldsymbol{z} \{1 + O(N^{-1/2}|\triangle|^{-1})\} \leq C.$$

Next note that for any $\ell, i, j \neq j'$, $E\{X_{i\ell}^2 \widetilde{\mathbf{B}}(\boldsymbol{z}_j)^\top \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \varepsilon_{ij} \varepsilon_{ij'}\} = 0$, and for any $\ell, i \neq i'$, $j, j', E\{X_{i\ell}X_{i'\ell}\widetilde{\mathbf{B}}(\boldsymbol{z}_j)^\top \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \sigma(\boldsymbol{z}_j) \sigma(\boldsymbol{z}_{j'}) \varepsilon_{ij} \varepsilon_{i'j'}\} = 0$. Therefore,

$$E\|\widehat{\boldsymbol{\theta}}_{\varepsilon}\|^{2} \asymp \frac{1}{nN|\Delta|^{4}} \sum_{\ell=0}^{p} E(X_{i\ell}^{2}) \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) \leq Cp(nN)^{-1}|\Delta|^{-4}.$$

The conclusion of the lemma follows.

Proof of Theorem 1. By Lemma A.11, Lemma A.12, and the properties of the bivariate spline basis functions in Lemma A.1, $\|\widehat{\eta}_{\ell}\|_{L_2}^2 \simeq |\Delta|^2 \|\widehat{\boldsymbol{\theta}}_{\eta,\ell}\|^2 = O_P(n^{-1})$ and $\|\widehat{\varepsilon}_{\ell}\|_{L_2}^2 \simeq |\Delta|^2 \|\widehat{\boldsymbol{\theta}}_{\varepsilon,\ell}\|^2 = O_P(n^{-1}N^{-1}|\Delta|^{-2})$, for any $\ell = 0, 1, \ldots, p$. It is clear that $\|\widehat{\beta}_{\ell} - \beta_{\ell}^o\|_{L_2}^2 \leq \|\widehat{\beta}_{\mu,\ell} - \beta_{\ell}^o\|_{L_2}^2 + \|\widehat{\eta}_{\ell}\|_{L_2}^2 + \|\widehat{\varepsilon}_{\ell}\|_{L_2}^2$, where the asymptotic order of $\|\widehat{\beta}_{\mu,\ell} - \beta_{\ell}^o\|_{L_2}$ is the same as $\|\widehat{\beta}_{\mu,\ell} - \beta_{\ell}^o\|_{\infty}$. The desired result follows from Lemma A.10. \square

Lemma A.13. Under Assumptions (A1)-(A6), if for any $\ell = 0, 1, ..., p$, $|X_{i\ell}| \leq C_{\ell} < \infty$, then as $N \to \infty$ and $n \to \infty$, one has for any vector $\mathbf{a} = (\mathbf{a}_0^{\top}, ..., \mathbf{a}_p^{\top})^{\top}$ with $\mathbf{a}^{\top} \mathbf{a} = 1$, $[\operatorname{Var}\{\mathbf{a}^{\top}(\widehat{\boldsymbol{\theta}}_{\eta} + \widehat{\boldsymbol{\theta}}_{\varepsilon})\}]^{-1/2}\{\mathbf{a}^{\top}(\widehat{\boldsymbol{\theta}}_{\eta} + \widehat{\boldsymbol{\theta}}_{\varepsilon})\} \stackrel{\mathcal{L}}{\longrightarrow} N(0, 1)$, where $\widehat{\boldsymbol{\theta}}_{\eta}$ and $\widehat{\boldsymbol{\theta}}_{\varepsilon}$ are given in (A.37).

Proof. For coefficient vectors $\widehat{\boldsymbol{\theta}}_{\eta}$, $\widehat{\boldsymbol{\theta}}_{\varepsilon}$ and the matrix $\Gamma_{n,\rho}$ defined in (A.17), $\operatorname{Var}\{\mathbf{a}^{\top}(\widehat{\boldsymbol{\theta}}_{\eta} + \widehat{\boldsymbol{\theta}}_{\varepsilon})\} = \mathbf{a}^{\top}\{E(\widehat{\boldsymbol{\theta}}_{\eta}\widehat{\boldsymbol{\theta}}_{\eta}^{\top}) + E(\widehat{\boldsymbol{\theta}}_{\varepsilon}\widehat{\boldsymbol{\theta}}_{\varepsilon}^{\top})\}\mathbf{a}$. Denote $\Psi_{\eta} = (\Psi_{\eta,\ell,\ell'})_{\ell,\ell'}$ and $\Psi_{\varepsilon} = (\Psi_{\epsilon,\ell,\ell'})_{\ell,\ell'}$, with

$$\boldsymbol{\Psi}_{\eta,\ell,\ell'} = \frac{1}{n^2 N^2} \sum_{i=1}^n \sum_{j=1}^N \sum_{j'=1}^N X_{i\ell} X_{i\ell'} \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \widetilde{\mathbf{B}}^\top(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}),$$

$$oldsymbol{\Psi}_{arepsilon,\ell,\ell'} = rac{1}{n^2N^2} \sum_{i=1}^n \sum_{j=1}^N X_{i\ell} X_{i\ell'} \widetilde{\mathbf{B}}(oldsymbol{z}_j) \widetilde{\mathbf{B}}^ op(oldsymbol{z}_j) \sigma^2(oldsymbol{z}_j),$$

then, we have

$$\mathbf{a}^{\top} E(\widehat{\boldsymbol{\theta}}_{\eta} \widehat{\boldsymbol{\theta}}_{\eta}^{\top}) \mathbf{a} = E \mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{n^{2} N^{2}} \sum_{i=1}^{n} \sum_{j,j'=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \right\}^{\top} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a}$$

$$= E \mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \boldsymbol{\Psi}_{\eta} \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a},$$

$$\mathbf{a}^{\top} E(\widehat{\boldsymbol{\theta}}_{\varepsilon} \widehat{\boldsymbol{\theta}}_{\varepsilon}^{\top}) \mathbf{a} = E \mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{n^{2} N^{2}} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\} \left\{ \widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \right\}^{\top} \sigma^{2}(\boldsymbol{z}_{j}) \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a}$$

$$= E \mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \boldsymbol{\Psi}_{\varepsilon} \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a}.$$

Note that for any vector \mathbf{a} with $\mathbf{a}^{\top}\mathbf{a} = 1$, we can rewrite as $\mathbf{a}^{\top}(\widehat{\boldsymbol{\theta}}_{\eta} + \widehat{\boldsymbol{\theta}}_{\varepsilon}) = \sum_{i=1}^{n} a_{i}^{\eta+\varepsilon} \boldsymbol{\mathfrak{z}}_{i}$, where

$$\begin{split} (a_i^{\eta+\varepsilon})^2 = &\mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{n^2 N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} \left\{ \widetilde{\mathbf{X}}_i \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \right\} \left\{ \widetilde{\mathbf{X}}_i \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'}) \right\}^{\top} G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a} \\ &+ \mathbf{a}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{n^2 N^2} \sum_{j=1}^{N} \left\{ \widetilde{\mathbf{X}}_i \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \right\} \left\{ \widetilde{\mathbf{X}}_i \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \right\}^{\top} \sigma^2(\boldsymbol{z}_j) \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{a} = (a_i^{\eta})^2 + (a_i^{\varepsilon})^2, \end{split}$$

and conditional on $\{\widetilde{\mathbf{X}}_i, i=1,\ldots,n\}$, \mathfrak{z}_i are independent with mean zero and variance one. Thus, $\sum_{i=1}^n (a_i^{\eta})^2 = \mathbf{a}^{\top} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{\Psi}_{\eta} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{a}$ and $\sum_{i=1}^n (a_i^{\varepsilon})^2 = \mathbf{a}^{\top} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{\Psi}_{\varepsilon} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{a}$.

According to Lemma A.9, Assumptions (A2) and (A4),

$$E\mathbf{a}^{\top}\boldsymbol{\Gamma}_{n,\rho}^{-1}\boldsymbol{\Psi}_{\eta}\boldsymbol{\Gamma}_{n,\rho}^{-1}\mathbf{a} \geq c_{\Gamma}^{-2}\left(|\Delta|^{2} + \frac{\rho_{n}}{nN|\Delta|^{2}}\right)^{-2}E\mathbf{a}^{\top}\boldsymbol{\Psi}_{\eta}\mathbf{a},$$

where

$$\mathbf{a}^{\top} \mathbf{\Psi}_{\eta} \mathbf{a} = \frac{1}{n^2 N^2} \sum_{\ell,\ell'=0}^{p} \sum_{i=1}^{n} \sum_{j=1}^{N} \sum_{j'=1}^{N} X_{i\ell} X_{i\ell'} \mathbf{a}_{\ell}^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \boldsymbol{a}_{\ell'} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'})$$

$$= \frac{1}{n^2} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \left\{ \frac{1}{N} \sum_{\ell=0}^{p} \sum_{j=1}^{N} \lambda_{k}^{1/2} X_{i\ell} g_{\ell}(\boldsymbol{z}_{j}) \psi_{k}(\boldsymbol{z}_{j}) \right\}^{2}$$

with $g_{\ell}(z) = a_{\ell}^{\top} \widetilde{\mathbf{B}}(z)$. Therefore, by Assumption (A3), we have

$$E\mathbf{a}^{\mathsf{T}}\mathbf{\Psi}_{\eta}\mathbf{a} = \frac{1}{n} \sum_{k=1}^{\infty} \sum_{\ell=0}^{p} \left\{ \frac{1}{N} \sum_{j=1}^{N} \lambda_{k}^{1/2} g_{\ell}(\boldsymbol{z}_{j}) \psi_{k}(\boldsymbol{z}_{j}) \right\}^{2}$$

$$\geq \frac{c}{nN^{2}} \sum_{\ell=0}^{p} \sum_{j=1}^{N} \sum_{j'=1}^{N} g_{\ell}(\boldsymbol{z}_{j}) g_{\ell}(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'})$$

$$\approx \frac{1}{n} \sum_{\ell=0}^{p} \int_{\Omega^{2}} g_{\ell}(\boldsymbol{z}) g_{\ell}(\boldsymbol{z}') G_{\eta}(\boldsymbol{z}, \boldsymbol{z}') d\boldsymbol{z} d\boldsymbol{z}'.$$

Noting that the eigenvalues of G_{η} are strictly positive, we have

$$E\mathbf{a}^{\mathsf{T}}\mathbf{\Psi}_{\eta}\mathbf{a} \geq c_1 n^{-1} \sum_{\ell=0}^{p} \int_{\Omega} g_{\ell}^{2}(\mathbf{z}) d\mathbf{z} \geq c_2 n^{-1} |\Delta|^2 ||\mathbf{a}||^2.$$

Therefore, we have $E\mathbf{a}^{\top}\mathbf{\Gamma}_{n,\rho}^{-1}\mathbf{\Psi}_{\eta}\mathbf{\Gamma}_{n,\rho}^{-1}\mathbf{a} \geq cn^{-1}\left(1+\frac{\rho_{n}}{nN|\triangle|^{4}}\right)^{-2}|\triangle|^{-2}$. Similarly, one can show that $E\mathbf{a}^{\top}\mathbf{\Gamma}_{n,\rho}^{-1}\mathbf{\Psi}_{\varepsilon}\mathbf{\Gamma}_{n,\rho}^{-1}\mathbf{a} \geq c(nN)^{-1}\left(1+\frac{\rho_{n}}{nN|\triangle|^{4}}\right)^{-2}|\triangle|^{-2}$. In addition,

$$\begin{aligned} \max(a_i^{\eta})^2 &\leq \frac{C}{|\triangle|^4} \mathbf{a}^{\top} \frac{1}{n^2 N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} \left\{ (\widetilde{\mathbf{X}}_i \widetilde{\mathbf{X}}_i^{\top}) \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \right\} G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) \mathbf{a} \\ &\leq \frac{C}{|\triangle|^4} \sum_{\ell,\ell'=1}^{p} \frac{1}{n^2 N^2} \max_{i} |X_{i\ell} X_{i\ell'}| \sum_{j=1}^{N} \sum_{j'=1}^{N} g_{\ell}(\boldsymbol{z}_j) g_{\ell'}(\boldsymbol{z}_{j'}) G_{\eta}(\boldsymbol{z}_j, \boldsymbol{z}_{j'}) \leq C n^{-2} |\triangle|^{-2}, \\ \max(a_i^{\varepsilon})^2 &\leq \frac{C}{|\triangle|^4} \mathbf{a}^{\top} \frac{1}{n^2 N^2} \sum_{j=1}^{N} \left\{ (\widetilde{\mathbf{X}}_i \widetilde{\mathbf{X}}_i^{\top}) \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \widetilde{\mathbf{B}}(\boldsymbol{z}_j)^{\top} \right\} \sigma^2(\boldsymbol{z}_j) \mathbf{a} \\ &\leq \frac{C}{|\triangle|^4} \sum_{\ell,\ell'=1}^{p} \frac{1}{n^2 N^2} \max_{i} |X_{i\ell} X_{i\ell'}| \sum_{j=1}^{N} g_{\ell}(\boldsymbol{z}_j) g_{\ell'}(\boldsymbol{z}_{j'}) \sigma^2(\boldsymbol{z}_j) \leq C n^{-2} N^{-1} |\triangle|^{-2}. \end{aligned}$$

Thus, if $\rho_n n^{-1} N^{-1} |\Delta|^{-4} \to 0$, we have

$$\frac{\max_{1 \le i \le n} (a_i^{\eta} + a_i^{\varepsilon})^2}{\sum_{i=1}^n (a_i^{\eta} + a_i^{\varepsilon})^2} \le Cn^{-1} \left(1 + \frac{\rho_n}{nN|\Delta|^4} \right)^2 \to 0,$$

which satisfies the Lindeberg condition.

Theorem A.6. Under Assumptions (A1)-(A6), if for any $\ell = 0, 1, ..., p$, $|X_{i\ell}| \leq C_{\ell} < \infty$, $\sup_{\boldsymbol{z} \in \Omega} [\operatorname{Var}\{\widehat{\beta}_{\ell}(\boldsymbol{z})\}]^{-1/2} (\widehat{\beta}_{\mu,\ell}(\boldsymbol{z}) - \beta_{\ell}^{o}(\boldsymbol{z})) = o_{P}(1)$, for $\ell = 0, ..., p$.

Proof. Using similar arguments as in the proof of Lemma A.13 and the result of Lemma A.9, we have for any $\|\mathbf{a}\| = 1$, $E\mathbf{a}^{\top} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{\Psi}_{\eta} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{a} \leq C_{\Gamma}^{-2} |\Delta|^{-4} E\mathbf{a}^{\top} \mathbf{\Psi}_{\eta} \mathbf{a} \leq Cn^{-1} |\Delta|^{-2}$, and $E\mathbf{a}^{\top} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{\Psi}_{\varepsilon} \mathbf{\Gamma}_{n,\rho}^{-1} \mathbf{a} \leq C_{\Gamma}^{-2} |\Delta|^{-4} E\mathbf{a}^{\top} \mathbf{\Psi}_{\varepsilon} \mathbf{a} \leq C(nN)^{-1} |\Delta|^{-2}$. Therefore, based on the proof of Lemma A.13, for any $\|\mathbf{a}\| = 1$,

$$cn^{-1}|\triangle|^{-4}\left(1+\frac{\rho_n}{nN|\triangle|^4}\right)^{-2} \leq E\mathbf{a}^{\mathsf{T}}\boldsymbol{\Gamma}_{n,\rho}^{-1}\boldsymbol{\Psi}_{\eta}\boldsymbol{\Gamma}_{n,\rho}^{-1}\mathbf{a} \leq Cn^{-1}|\triangle|^{-2},$$

$$c(nN)^{-1}\left(1+\frac{\rho_n}{nN|\triangle|^4}\right)^{-2}|\triangle|^{-2} \leq E\mathbf{a}^{\mathsf{T}}\boldsymbol{\Gamma}_{n,\rho}^{-1}\boldsymbol{\Psi}_{\varepsilon}\boldsymbol{\Gamma}_{n,\rho}^{-1}\mathbf{a} \leq C(nN)^{-1}|\triangle|^{-2}.$$

Thus,

$$\operatorname{Var}(\widehat{\beta}_{\ell}) = \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} E\{\boldsymbol{\Gamma}_{n,\rho}^{-1}(\boldsymbol{\Psi}_{\eta} + \boldsymbol{\Psi}_{\varepsilon})\boldsymbol{\Gamma}_{n,\rho}^{-1}\}\{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}
\times \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} E(\boldsymbol{\Gamma}_{n,\rho}^{-1}\boldsymbol{\Psi}_{\eta}\boldsymbol{\Gamma}_{n,\rho}^{-1})\{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\} \times \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top}
\times E\left[\boldsymbol{\Gamma}_{n,\rho}^{-1} \frac{1}{n^{2}N^{2}} \sum_{i=1}^{n} \sum_{j,j'=1}^{N} \left\{\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\right\} \left\{\widetilde{\mathbf{X}}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right\}^{\top} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}) \boldsymbol{\Gamma}_{n,\rho}^{-1}\right] \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}.$$

By Lemma A.9, we have

$$\operatorname{Var}(\widehat{\beta}_{\ell}) \lesssim \frac{1}{nN^{2}|\Delta|^{2}} \sum_{j=1}^{N} \sum_{j'=1}^{N} \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'}),$$

$$\operatorname{Var}(\widehat{\beta}_{\ell}) \gtrsim \frac{1}{nN^{2}|\Delta|^{2}} \sum_{j=1}^{N} \sum_{j'=1}^{N} \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\}^{\top} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \{\mathbf{e}_{\ell} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z})\} G_{\eta}(\boldsymbol{z}_{j}, \boldsymbol{z}_{j'})$$

$$\times \left(1 + \frac{\rho_{n}}{nN|\Delta|^{4}}\right)^{-2},$$

and according to Lemmas A.1 and A.4, we have $cn^{-1}\left(1+\frac{\rho_n}{nN|\Delta|^4}\right)^{-2} \leq \operatorname{Var}(\widehat{\beta}_{\ell}) \leq Cn^{-1}$. According to Lemma A.10, if $\rho_n n^{-1/2} N^{-1} |\Delta|^{-3} \to 0$ and $n^{1/2} |\Delta|^{d+1} \to 0$, the bias term in (A.19) is negligible compared to the order of $[\operatorname{Var}\{\widehat{\beta}_{\ell}(\boldsymbol{z})\}]^{1/2}$.

Proof of Theorem 2. Theorem 2 follows from (A.19), Lemma A.13 and Theorem A.6. \Box

A.4. Asymptotic properties of piecewise constant spline estimators

In this section, we study the asymptotic properties of the piecewise constant spline estimators defined in the spline space $\mathcal{PC}(\Delta)$. Define piecewise constant bivariate spline

functions

$$\widehat{\boldsymbol{\beta}}_{\mu}^{c}(\boldsymbol{z}) = (\widehat{\boldsymbol{\beta}}_{\mu,0}^{c}(\boldsymbol{z}), \dots, \widehat{\boldsymbol{\beta}}_{\mu,p}^{c}(\boldsymbol{z}))^{\top} = \widehat{\mathbf{V}}_{m(\boldsymbol{z})}^{-1} \left\{ \frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) X_{i\ell} \sum_{\ell'=0}^{p} \boldsymbol{\beta}_{\ell'}^{o}(\boldsymbol{z}_{j}) X_{i\ell'} \right\}_{\ell=0}^{p},$$
(A.27)

$$\widehat{\boldsymbol{\eta}}(\boldsymbol{z}) = (\widehat{\eta}_0(\boldsymbol{z}), \dots, \widehat{\eta}_p(\boldsymbol{z}))^{\top} = \widehat{\mathbf{V}}_{m(\boldsymbol{z})}^{-1} \left\{ \frac{1}{nN} \sum_{i=1}^n \sum_{j=1}^N B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \sum_{k=1}^\infty \xi_{ik} \psi_k(\boldsymbol{z}_j) \right\}_{\ell=0}^p,$$
(A.28)

$$\widehat{\boldsymbol{\varepsilon}}(\boldsymbol{z}) = (\widehat{\varepsilon}_0(\boldsymbol{z}), \dots, \widehat{\varepsilon}_p(\boldsymbol{z}))^{\top} = \widehat{\mathbf{V}}_{m(\boldsymbol{z})}^{-1} \left\{ \frac{1}{nN} \sum_{i=1}^n \sum_{j=1}^N B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \varepsilon_{ij} \right\}_{\ell=0}^p, \tag{A.29}$$

where $\widehat{\mathbf{V}}_{m(z)}$ is defined in (2.7).

The next two theorems concern the functions $\widehat{\beta}_{\mu,\ell}^{c}(\boldsymbol{z})$, $\widehat{\eta}_{\ell}(\boldsymbol{z})$, $\widehat{\varepsilon}_{\ell}(\boldsymbol{z})$, $\ell = 0, \ldots, p$, given in (A.27), (A.28) and (A.29). Theorem A.7 gives the uniform convergence rate of $\widehat{\beta}_{\mu,\ell}(\boldsymbol{z})$ to $\beta_{\ell}^{o}(\boldsymbol{z})$.

Theorem A.7. Under Assumptions (A1'), (A2)-(A6), the constant spline functions $\widehat{\beta}_{\mu,\ell}^{c}(\boldsymbol{z}), \ \ell = 0, \ldots, p, \ satisfy \ \sup_{\boldsymbol{z} \in \Omega} \sup_{0 \le \ell \le p} \left| \widehat{\beta}_{\mu,\ell}^{c}(\boldsymbol{z}) - \beta_{\ell}^{o}(\boldsymbol{z}) \right| = O_{P}(|\Delta|).$

In the following, we provide detailed proofs of Theorems A.7. For the random matrix $\widehat{\mathbf{V}}_m$ defined in (2.7), the lemma below shows that its inverse can be approximated by the inverse of a deterministic matrix $A_m^{-1} \Sigma_X^{-1}$, where $A_m = \int_{\Omega} B_m(\boldsymbol{z}) d\boldsymbol{z}$.

Lemma A.14. Under Assumptions (A3) and (A5), for any $m \in \mathcal{M}$, we have

$$\widehat{\mathbf{V}}_{m}^{-1} = A_{m}^{-1} \Sigma_{X}^{-1} + O_{P} \left\{ n^{-1/2} |\Delta|^{2} (\log n)^{1/2} + N^{-1/2} |\Delta| \right\}. \tag{A.30}$$

Proof. By Lemma A.5, $\|\widehat{\mathbf{V}}_m - A_m \mathbf{\Sigma}_X\|_{\infty} = O_P \{n^{-1/2} |\Delta|^2 (\log n)^{1/2} + N^{-1/2} |\Delta| \}$. Using the fact that for any matrices \mathbf{A} and \mathbf{B} , $(\mathbf{A} + \delta \mathbf{B})^{-1} = \mathbf{A}^{-1} - \delta \mathbf{A}^{-1} \mathbf{B} \mathbf{A}^{-1} + O(\delta^2)$, we obtain (A.30).

Proof of Theorem A.7. According to Lemma A.2, there exist functions $\beta_{\ell}^* \in \mathcal{PC}(\Delta)$ that satisfies $\|\beta_{\ell}^* - \beta_{\ell}^o\|_{\infty} = O(|\Delta|)$ for $\ell = 0, 1, ..., p$. By the definition of $\widehat{\beta}_{\mu,\ell}(\boldsymbol{z})$ in (A.27), $\widehat{\boldsymbol{\beta}}_{\mu}^{c}(\boldsymbol{z}) = \left(\widehat{\beta}_{\mu,0}^{c}(\boldsymbol{z}), \widehat{\beta}_{\mu,1}^{c}(\boldsymbol{z}), ..., \widehat{\beta}_{\mu,p}^{c}(\boldsymbol{z})\right)^{\top} = \left(\widetilde{\gamma}_{m(\boldsymbol{z}),0}, ..., \widetilde{\gamma}_{m(\boldsymbol{z}),p}\right)^{\top} = \widetilde{\boldsymbol{\gamma}}_{m(\boldsymbol{z})}$, where $\widetilde{\boldsymbol{\gamma}}_{m} = \widehat{\mathbf{V}}_{m}^{-1} \left\{ (nN)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m}(\boldsymbol{z}_{j}) X_{i\ell} \sum_{\ell=0}^{p} \beta_{\ell'}^{o}(\boldsymbol{z}_{j}) X_{i\ell'} \right\}_{\ell=0}^{p}$ for $\widehat{\mathbf{V}}_{m}$ defined in (2.7). Let

$$\widetilde{\boldsymbol{\beta}}(\boldsymbol{z}) = (\widetilde{\beta}_0(\boldsymbol{z}), \widetilde{\beta}_1(\boldsymbol{z}), \dots, \widetilde{\beta}_p(\boldsymbol{z}))^{\top} = \widehat{\mathbf{V}}_{m(\boldsymbol{z})}^{-1} \left[\frac{1}{nN} \sum_{i=1}^n \sum_{j=1}^N B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \sum_{\ell'=0}^p \beta_{\ell'}^*(\boldsymbol{z}_j) X_{i\ell'} \right]_{\ell=0}^p,$$

then

$$\widehat{\boldsymbol{\beta}}_{\mu}^{c}(\boldsymbol{z}) - \widetilde{\boldsymbol{\beta}}(\boldsymbol{z}) = \widehat{\mathbf{V}}_{m(\boldsymbol{z})}^{-1} \left[\frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) X_{i\ell} \sum_{\ell'=0}^{p} \left\{ \beta_{\ell'}^{o}(\boldsymbol{z}_{j}) - \beta_{\ell'}^{*}(\boldsymbol{z}_{j}) \right\} X_{i\ell'} \right]_{\ell=0}^{p}.$$

Observing that $\widetilde{\beta}_{\ell} \equiv \beta_{\ell}^*$ as $\beta_{\ell}^* \in \mathcal{PC}(\Delta)$, $\widehat{\beta}_{\mu,\ell}^c(\boldsymbol{z}) = \widehat{\beta}_{\mu,\ell}^c(\boldsymbol{z}) - \widetilde{\beta}_{\ell}(\boldsymbol{z}) + \beta_{\ell}^*(\boldsymbol{z})$, $\ell = 0, 1, \dots, p$. It is easy to see $\|\widehat{\beta}_{\mu,\ell}^c - \widetilde{\beta}_{\ell}\|_{\infty} = O_P(|\Delta|)$. Hence, for $\ell = 0, 1, \dots, p$, $\|\widehat{\beta}_{\mu,\ell}^c - \beta_{\ell}^o\|_{\infty} \leq \|\widehat{\beta}_{\mu,\ell}^c - \widetilde{\beta}_{\ell}\|_{\infty} + \|\beta_{\ell}^o - \beta_{\ell}^*\|_{\infty} = O_P(|\Delta|)$, which completes the proof.

By Lemma A.14, the inverse of the random matrix $\hat{\mathbf{V}}_m$ can be approximated by that of a deterministic matrix $A_m \Sigma_X$. Substituting $\hat{\mathbf{V}}_m$ with $A_m \Sigma_X$ in (A.28) and (A.29), we define the random vectors

$$\widehat{\boldsymbol{\eta}}^*(\boldsymbol{z}) = (\widehat{\eta}_0^*(\boldsymbol{z}), \dots, \widehat{\eta}_p^*(\boldsymbol{z}))^{\top} = A_{m(\boldsymbol{z})}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \sum_{k=1}^{\infty} \xi_{ik} \psi_k(\boldsymbol{z}_j) \bigg\}_{\ell=0}^{p},$$
(A.31)

$$\widehat{\boldsymbol{\varepsilon}}^*(\boldsymbol{z}) = (\widehat{\varepsilon}_0^*(\boldsymbol{z}), \dots, \widehat{\varepsilon}_p^*(\boldsymbol{z}))^{\top} = A_{m(\boldsymbol{z})}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \varepsilon_{ij} \bigg\}_{\ell=0}^{p}.$$
(A.32)

The next lemma implies that the difference between $\widehat{\eta}^*(z)$ and $\widehat{\eta}(z)$ and the difference between $\widehat{\varepsilon}^*(z)$ and $\widehat{\varepsilon}(z)$ are both negligible uniformly over $z \in \Omega$.

Lemma A.15. Under Assumptions (A2)-(A5) and (C1), if $N^{1/2}|\Delta| \to \infty$ as $N \to \infty$, $\|\sum_{k=1}^{\infty} \lambda_k^{1/2} \psi_k\|_{\infty} < \infty$ and $n^{1/(4+\delta_2)} \ll n^{1/2} N^{-1/2} |\Delta|^{-1}$ for some δ_2 , then for $\widehat{\boldsymbol{\eta}}(\boldsymbol{z})$, $\widehat{\boldsymbol{\varepsilon}}(\boldsymbol{z})$ given in (A.28), (A.29) and $\widehat{\boldsymbol{\eta}}^*(\boldsymbol{z})$, $\widehat{\boldsymbol{\varepsilon}}^*(\boldsymbol{z})$ given in (A.31), (A.32), as $N \to \infty$ and $n \to \infty$, we have

$$\sup_{z \in \Omega} \|\widehat{\boldsymbol{\eta}}(z) - \widehat{\boldsymbol{\eta}}^*(z)\|_{\infty} = O_P \left\{ n^{-1} |\Delta|^4 \log(n) + n^{-1/2} N^{-1/2} |\Delta|^3 (\log n)^{1/2} \right\}, \tag{A.33}$$

$$\sup_{\boldsymbol{z} \in \Omega} \|\widehat{\boldsymbol{\varepsilon}}(\boldsymbol{z}) - \widehat{\boldsymbol{\varepsilon}}^*(\boldsymbol{z})\|_{\infty} = O_P \left\{ n^{-1} N^{-1/2} |\Delta|^3 \log(n) + n^{-1/2} N^{-1} |\Delta|^2 (\log n)^{1/2} \right\}. \quad (A.34)$$

Proof. Comparing $\widehat{\boldsymbol{\eta}}(\boldsymbol{z})$ and $\widehat{\boldsymbol{\eta}}^*(\boldsymbol{z})$ given in (A.28) and (A.31), we have

$$\widehat{\boldsymbol{\eta}}(\boldsymbol{z}) - \widehat{\boldsymbol{\eta}}^*(\boldsymbol{z}) = \left\{ \widehat{\mathbf{V}}_m^{-1} - A_{m(\boldsymbol{z})}^{-1} \boldsymbol{\Sigma}_X^{-1} \right\} \left\{ \frac{1}{nN} \sum_{i=1}^n \sum_{j=1}^N B_{m(\boldsymbol{z})}(\boldsymbol{z}_j) X_{i\ell} \sum_{k=1}^\infty \xi_{ik} \psi_k(\boldsymbol{z}_j) \right\}_{\ell=0}^p.$$

Now let $\zeta_{i,m,\ell} \equiv \zeta_i = n^{-1} \left[X_{i\ell} \sum_{k=1}^{\infty} \left\{ \frac{1}{N} \sum_{j=1}^{N} B_m(\boldsymbol{z}_j) \psi_k(\boldsymbol{z}_j) \right\} \xi_{ik} \right]$, then it is easy to see that $\frac{1}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_m(\boldsymbol{z}_j) X_{i\ell} \sum_{k=1}^{\infty} \xi_{ik} \psi_k(\boldsymbol{z}_j) = \frac{1}{N} \sum_{i=1}^{n} \zeta_{i,m,\ell}$. It is easy to see that $E(\zeta_i) = 0$, and

$$\sigma_{\zeta_{i},n}^{2} = E\left(\zeta_{i}^{2}\right) = n^{-2}E(X_{\ell}^{2}) \int_{T_{m}\times T_{m}} G_{\eta}\left(\boldsymbol{u},\boldsymbol{v}\right) d\boldsymbol{u} d\boldsymbol{v} \{1 + O(N^{-1/2}|\triangle|^{-1})\}.$$

Note that $\left\{\sigma_{\zeta_i,n}^{-1}\zeta_i\right\}_{i=1}^n$ are uncorrelated random variables with mean 0. Assume that $|\Delta|^{-2} \asymp n^{\tau}$ for some $0 < \tau < \infty$, we can show that for any large enough $\delta > 0$, $P\left[\left|\sum_{i=1}^n \zeta_i\right| \ge \delta \left\{C \log(n) n^{-1} |\Delta|^4 E(X_{i\ell}^2)\right\}^{1/2}\right] \le 2n^{-2-\tau}$. Therefore,

$$\sum_{n=1}^{\infty} P\left\{ \sup_{m \in \mathcal{M}, 0 \le \ell \le p} \left| \sum_{i=1}^{n} \zeta_{i,m,\ell} \right| \ge \delta n^{-1/2} |\Delta|^2 (\log n)^{1/2} \right\} < \infty.$$

Thus, $\sup_{m,\ell} |\sum_{i=1}^n \zeta_{i,m,\ell}| = O_P \left\{ n^{-1/2} |\Delta|^2 (\log n)^{1/2} \right\}$ as $n \to \infty$ by Borel-Cantelli Lemma. It follows that $\sup_{m,\ell} |n^{-1} \sum_{i=1}^n \zeta_{i,m,\ell}| = O_P \left\{ n^{-1/2} |\Delta|^2 (\log n)^{1/2} \right\}$. Finally, according to (A.30), we obtain (A.33). The result in (A.34) can be proved similarly. \square

Lemma A.16. For any $z \in \Omega$, the covariance matrices of $\widehat{\eta}^*(z)$ and $\widehat{\varepsilon}^*(z)$ are

$$\Sigma_{\eta}(\boldsymbol{z}) = E\left\{\widehat{\boldsymbol{\eta}}^{*}(\boldsymbol{z})\widehat{\boldsymbol{\eta}}^{*\top}(\boldsymbol{z})\right\} = A_{m(\boldsymbol{z})}^{-2}\Sigma_{X}^{-1}\frac{1}{nN^{2}}\sum_{k=1}^{\infty}\lambda_{k}\left\{\sum_{j=1}^{N}B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j})\psi_{k}(\boldsymbol{z}_{j})\right\}^{2},$$

$$\Sigma_{\varepsilon}(\boldsymbol{z}) = E\left\{\widehat{\boldsymbol{\varepsilon}}^{*}(\boldsymbol{z})\widehat{\boldsymbol{\varepsilon}}^{*\top}(\boldsymbol{z})\right\} = A_{m(\boldsymbol{z})}^{-2}\Sigma_{X}^{-1}\frac{1}{nN^{2}}\sum_{j=1}^{N}B_{m(\boldsymbol{z})}^{2}(\boldsymbol{z}_{j})\sigma^{2}(\boldsymbol{z}_{j}),$$

in addition,

$$\sup_{\boldsymbol{z} \in \Omega} \|\boldsymbol{\Sigma}_{\eta}(\boldsymbol{z}) + \boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{z}) - \boldsymbol{\Sigma}_{n}(\boldsymbol{z})\|_{\infty} = O(n^{-1}N^{-1/2}|\Delta|^{-1}), \tag{A.35}$$

where $\Sigma_n(z)$ is given in (2.9).

Proof. Note that $A_{m(\boldsymbol{z})}^2 \widehat{\boldsymbol{\eta}}^*(\boldsymbol{z}) \widehat{\boldsymbol{\eta}}^{*\top}(\boldsymbol{z})$ is equal to

$$\Sigma_{X}^{-1} \left\{ \frac{1}{n^{2}N^{2}} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) X_{i\ell} \sum_{k=1}^{\infty} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \sum_{i'=1}^{n} \sum_{j'=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j'}) X_{i'\ell'} \sum_{k'=1}^{\infty} \xi_{i'k'} \psi_{k'}(\boldsymbol{z}_{j'}) \right\}_{\ell,\ell'=0}^{p} \Sigma_{X}^{-1}.$$

Thus,

$$\boldsymbol{\Sigma}_{\eta}(\boldsymbol{z}) = E\left\{\widehat{\boldsymbol{\eta}}^{*}(\boldsymbol{z})\widetilde{\boldsymbol{\eta}}^{\top}(\boldsymbol{z})\right\} = A_{m(\boldsymbol{z})}^{-2}\boldsymbol{\Sigma}_{X}^{-1}\frac{1}{nN^{2}}\sum_{k=1}^{\infty}\lambda_{k}\left\{\sum_{j=1}^{N}B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j})\psi_{k}(\boldsymbol{z}_{j})\right\}^{2}.$$

Similarly, we can derive the covariance of $\hat{\boldsymbol{\varepsilon}}^*(\boldsymbol{z})$: $\boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{z}) = A_{m(\boldsymbol{z})}^{-2} \boldsymbol{\Sigma}_X^{-1} \frac{1}{nN^2} \sum_{j=1}^N B_{m(\boldsymbol{z})}^2(\boldsymbol{z}_j) \sigma^2(\boldsymbol{z}_j)$. Observe that

$$\sum_{k=1}^{\infty} \lambda_k \left\{ \frac{1}{N} \sum_{j=1}^{N} B_{m(z)}(z_j) \psi_k(z_j) \right\}^2 = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{j'=1}^{N} G_{\eta}(z_j, z_{j'}) B_{m(z)}(z_j) B_{m(z)}(z_{j'}).$$

Hence, by (A.5) and (A.6) in Lemma A.4, (A.35) holds. Therefore,

$$\Sigma_{\eta}(z) + \Sigma_{\varepsilon}(z) = (nA_{m(z)}^{2})^{-1} \Sigma_{X}^{-1} \int_{T_{m(z)} \times T_{m(z)}} G_{\eta}(\boldsymbol{u}, \boldsymbol{v}) d\boldsymbol{u} d\boldsymbol{v} \{ 1 + O(N^{-1/2} |\Delta|^{-1}) \}$$

$$+ (nNA_{m(z)}^{2})^{-1} \Sigma_{X}^{-1} \int_{T_{m(z)}} \sigma^{2}(\boldsymbol{u}) d\boldsymbol{u} \{ 1 + O(N^{-1/2} |\Delta|^{-1}) \}$$

$$= n^{-1} \Sigma_{X}^{-1} G_{\eta}(\boldsymbol{z}, \boldsymbol{z}) \{ 1 + O(N^{-1/2} |\Delta|^{-1}) \}.$$

Therefore, $\sup_{\boldsymbol{z}\in\Omega} \|\boldsymbol{\Sigma}_{\eta}(\boldsymbol{z}) + \boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{z}) - n^{-1}\boldsymbol{\Sigma}_{X}^{-1}G_{\eta}(\boldsymbol{z},\boldsymbol{z})\|_{\infty} = O(n^{-1}N^{-1/2}|\Delta|^{-1})$. The desired result in (A.35) follows.

Proof of Theorem 3. Note that, for any vector $\mathbf{a} = (a_0, \dots, a_p)^{\top} \in \mathcal{R}^{(p+1)}$, we have $E\left[\sum_{\ell=0}^{p} a_{\ell} \left\{ \widehat{\eta}_{\ell}^{*}(\boldsymbol{z}) + \widehat{\varepsilon}_{\ell}^{*}(\boldsymbol{z}) \right\} \right] = 0$, and

$$\sum_{\ell=0}^{p} a_{\ell} \widehat{\eta}_{\ell}^{*}(\boldsymbol{z}) = \boldsymbol{a}^{\top} \frac{A_{m(\boldsymbol{z})}^{-1} \boldsymbol{\Sigma}_{X}^{-1}}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) \sum_{k=1}^{\infty} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \mathbf{X}_{i} = \sum_{i=1}^{n} \boldsymbol{a}^{\top} A_{m(\boldsymbol{z})}^{-1} \boldsymbol{\Sigma}_{X}^{-1} \boldsymbol{\mathfrak{z}}_{i}^{\eta},$$

$$\sum_{\ell=0}^{p} a_{\ell} \widehat{\varepsilon}_{\ell}^{*}(\boldsymbol{z}) = \boldsymbol{a}^{\top} \frac{A_{m(\boldsymbol{z})}^{-1} \boldsymbol{\Sigma}_{X}^{-1}}{nN} \sum_{i=1}^{n} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) \varepsilon_{ij} \mathbf{X}_{i} = \sum_{i=1}^{n} \boldsymbol{a}^{\top} A_{m(\boldsymbol{z})}^{-1} \boldsymbol{\Sigma}_{X}^{-1} \boldsymbol{\mathfrak{z}}_{i}^{\varepsilon},$$

where $\boldsymbol{\mathfrak{z}}_{i}^{\eta} = \frac{1}{nN} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) \sum_{k=1}^{\infty} \xi_{ik} \psi_{k}(\boldsymbol{z}_{j}) \boldsymbol{X}_{i}$ and $\boldsymbol{\mathfrak{z}}_{i}^{\varepsilon} = \frac{1}{nN} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) \varepsilon_{ij} \boldsymbol{X}_{i}$ are independent sequences with variances $\operatorname{Var}(\boldsymbol{\mathfrak{z}}_{i}^{\eta}) = \frac{1}{n^{2}N^{2}} \sum_{j,j'} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}') \boldsymbol{\Sigma}_{X}$ and $\operatorname{Var}(\boldsymbol{\mathfrak{z}}_{i}^{\varepsilon}) = \frac{1}{n^{2}N^{2}} \sum_{j=1}^{N} B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j}) \sigma^{2}(\boldsymbol{z}_{j}) \boldsymbol{\Sigma}_{X}$, respectively. Therefore, we have

$$\operatorname{Var}\left(\boldsymbol{a}^{\top}A_{m(\boldsymbol{z})}^{-1}\boldsymbol{\Sigma}_{X}^{-1}\boldsymbol{\mathfrak{z}}_{i}^{\eta}\right) = \frac{1}{n}\boldsymbol{a}^{\top}\boldsymbol{\Sigma}_{\eta}(\boldsymbol{z})\boldsymbol{a} = \frac{A_{m(\boldsymbol{z})}^{-2}}{n^{2}N^{2}}\sum_{j,j'=1}^{N}B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j})B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j'})G_{\eta}(\boldsymbol{z}_{j},\boldsymbol{z}_{j'})\boldsymbol{a}^{\top}\boldsymbol{\Sigma}_{X}^{-1}\boldsymbol{a},$$

$$\operatorname{Var}\left(\boldsymbol{a}^{\top}A_{m(\boldsymbol{z})}^{-1}\boldsymbol{\Sigma}_{X}^{-1}\boldsymbol{\mathfrak{z}}_{i}^{\varepsilon}\right) = \frac{1}{n}\boldsymbol{a}^{\top}\boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{z})\boldsymbol{a} = \frac{A_{m(\boldsymbol{z})}^{-2}}{n^{2}N^{2}}\sum_{j=1}^{N}B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j})B_{m(\boldsymbol{z})}(\boldsymbol{z}_{j})\sigma^{2}(\boldsymbol{z}_{j})\boldsymbol{a}^{\top}\boldsymbol{\Sigma}_{X}^{-1}\boldsymbol{a}.$$

Using central limit theorem, we have

$$\left[\boldsymbol{a}^{\top}\left\{\boldsymbol{\Sigma}_{\eta}(\boldsymbol{z}) + \boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{z})\right\}\boldsymbol{a}\right]^{-1/2} \sum_{\ell=0}^{p} a_{\ell}\left\{\widehat{\eta}_{\ell}^{*}(\boldsymbol{z}) + \widehat{\varepsilon}_{\ell}^{*}(\boldsymbol{z})\right\} \stackrel{\mathcal{L}}{\longrightarrow} N(0,1).$$

By (A.35), as $N \to \infty$ and $n \to \infty$, $\{\mathbf{a}^{\top} \mathbf{\Sigma}_{n}(\mathbf{z}) \mathbf{a}\}^{-1/2} \sum_{\ell=0}^{p} a_{\ell} \{\widehat{\eta}_{\ell}^{*}(\mathbf{z}) + \widehat{\varepsilon}_{\ell}^{*}(\mathbf{z})\} \xrightarrow{\mathcal{L}} N(0,1)$. Therefore, $\{\mathbf{a}^{\top} \mathbf{\Sigma}_{n}(\mathbf{z}) \mathbf{a}\}^{-1/2} \sum_{\ell=0}^{p} a_{\ell} \{\widehat{\beta}_{\ell}^{c}(\mathbf{z}) - \beta_{\ell}^{o}(\mathbf{z})\} \xrightarrow{\mathcal{L}} N(0,1)$ follows from (A.18), Theorem A.7, Lemma A.15 and Slutsky's Theorem. Applying Cramér-Wold's device, we obtain $\mathbf{\Sigma}_{n}^{-1/2}(\mathbf{z}) \{\widehat{\beta}_{\ell}^{c}(\mathbf{z}) - \beta_{\ell}^{o}(\mathbf{z})\}_{\ell=0}^{p} \xrightarrow{\mathcal{L}} N(\mathbf{0}, \mathbf{I}_{(p+1)\times(p+1)})$, as $N \to \infty$ and $n \to \infty$, and consequently, $\sigma_{n,\ell\ell}^{-1}(\mathbf{z}) \{\widehat{\beta}_{\ell}^{c}(\mathbf{z}) - \beta_{\ell}^{o}(\mathbf{z})\} \xrightarrow{\mathcal{L}} N(0,1)$, for any $\mathbf{z} \in \Omega$ and $\ell = 0, \ldots, p$.

A.5. Convergence of the covariance estimator

For any i = 1, ..., n, and estimated residuals $\widehat{R}_{ij} = Y_{ij} - \sum_{\ell=0}^{p} X_{i\ell} \widehat{\beta}_{\ell}(\boldsymbol{z}_{j})$, denote $\widehat{\boldsymbol{\vartheta}}_{i} = \arg\min_{\boldsymbol{\theta}} \sum_{j=1}^{N} \left\{ \widehat{R}_{ij} - \mathbf{B}_{\eta}^{\top}(\boldsymbol{z}_{j}) \mathbf{Q}_{\eta,2} \boldsymbol{\theta} \right\}^{2}$, where $\mathbf{B}_{\eta}(\boldsymbol{z})$ is the set of bivariate spline basis functions used to estimate $\eta_{i}(\boldsymbol{z})$, and $\mathbf{Q}_{\eta,2}$ is given in the following QR decomposition of the transpose of the smoothness matrix \mathbf{H}_{η} : $\mathbf{H}_{\eta}^{\top} = \mathbf{Q}_{\eta} \mathbf{R}_{\eta} = (\mathbf{Q}_{\eta,1} \mathbf{Q}_{\eta,2}) \begin{pmatrix} \mathbf{R}_{\eta,1} \\ \mathbf{R}_{\eta,2} \end{pmatrix}$. Then, the bivariate spline estimator of $\eta_{i}(\boldsymbol{z})$ can be written as $\widehat{\eta}_{i}(\boldsymbol{z}) = \mathbf{B}_{\eta}(\boldsymbol{z})^{\top} \mathbf{Q}_{\eta,2} \widehat{\boldsymbol{\vartheta}}_{i} = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \widehat{\boldsymbol{\vartheta}}_{i}$. Let

$$\boldsymbol{\Upsilon}_n = \frac{1}{N} \sum_{j=1}^N \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_j) \widetilde{\mathbf{B}}_{\eta}^{\top}(\boldsymbol{z}_j),$$

then we have

$$\widehat{\boldsymbol{\vartheta}}_{i} = \boldsymbol{\Upsilon}_{n}^{-1} \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j}) \widehat{R}_{ij}$$

$$= \boldsymbol{\Upsilon}_{n}^{-1} \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j}) \left[\sum_{\ell=0}^{p} X_{i\ell} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\ell}(\boldsymbol{z}_{j}) \} + \eta_{i}(\boldsymbol{z}_{j}) + \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij} \right]. \tag{A.36}$$

Lemma A.17. Under Assumptions (A3)-(A5), if $(N^{1/2}|\triangle_{\eta}|)/\log(|\triangle_{\eta}|^{-1}) \to \infty$ as $N \to \infty$, then there exist constants $0 < c_{\Upsilon} < C_{\Upsilon} < \infty$, such that with probability approaching 1 as $N \to \infty$, $n \to \infty$, $c_{\Upsilon}|\triangle_{\eta}|^2 \le \lambda_{\min}(\Upsilon_n) \le \lambda_{\max}(\Upsilon_n) \le C_{\Upsilon}|\triangle_{\eta}|^2$.

The proof is similar to the proof of A.7, thus omitted.

Next we define

$$\widetilde{b}_{i}(\boldsymbol{z}) = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j}) \sum_{\ell=0}^{p} X_{i\ell} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\ell}(\boldsymbol{z}_{j}) \}, \tag{A.37}$$

$$\widetilde{\eta}_i(\boldsymbol{z}) = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_n^{-1} \frac{1}{N} \sum_{j=1}^N \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_j) \eta_i(\boldsymbol{z}_j), \ \widetilde{\varepsilon}_i(\boldsymbol{z}) = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_n^{-1} \frac{1}{N} \sum_{j=1}^N \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_j) \sigma(\boldsymbol{z}_j) \varepsilon_{ij}.$$

Then, the estimation error $D_i(z) = \widehat{\eta}_i(z) - \eta_i(z)$ in (3.1) can be decomposed as the following:

$$D_i(\boldsymbol{z}) = \widetilde{b}_i(\boldsymbol{z}) + \nabla \eta_i(\boldsymbol{z}) + \widetilde{\varepsilon}_i(\boldsymbol{z}).$$

For any $z, z' \in \Omega$, denote

$$\widetilde{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}') = n^{-1} \sum_{i=1}^{n} \eta_{i}(\boldsymbol{z}) \eta_{i}(\boldsymbol{z}').$$

The following lemma shows the uniform convergence of $\widetilde{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}')$ to $G_{\eta}(\boldsymbol{z}, \boldsymbol{z}')$ in probability over all $(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2$.

Lemma A.18. Under Assumptions (A1)-(A5) and (C1)-(C3), $\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2} |\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}')| = O_P\{n^{-1/2}(\log n)^{1/2}\}.$

Proof. Let $\bar{\xi}_{\cdot kk'} = n^{-1} \sum_{i=1}^n \xi_{ik} \xi_{ik'}$, then

$$\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}') = \sum_{k=1}^{\infty} \lambda_{k} \psi_{k}(\boldsymbol{z}) \psi_{k}(\boldsymbol{z}') \left(\bar{\xi}_{\cdot kk} - 1\right) + \sum_{k \neq k'} \bar{\xi}_{\cdot kk'} (\lambda_{k} \lambda_{k'})^{1/2} \psi_{k}(\boldsymbol{z}) \psi_{k'}(\boldsymbol{z}').$$

As $E\left[\sum_{k=1}^{\infty} \lambda_k \psi_k(\boldsymbol{z}) \psi_k(\boldsymbol{z}') \left(\bar{\xi}_{\cdot kk} - 1\right)\right] = 0$, then $E\{\widetilde{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}') - G_{\eta}(\boldsymbol{z}, \boldsymbol{z}')\} = 0$. Note that $E\{\eta^2(\boldsymbol{z})\eta^2(\boldsymbol{z}')\} = G_{\eta}(\boldsymbol{z}, \boldsymbol{z})G_{\eta}(\boldsymbol{z}', \boldsymbol{z}') + 2G_{\eta}^2(\boldsymbol{z}, \boldsymbol{z}') + \sum_{k=1}^{\infty} \lambda_k^2 E(\xi_{1k}^4 - 3)\psi_k^2(\boldsymbol{z})\psi_k^2(\boldsymbol{z}')$. Next,

$$E\left\{\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}')\right\}^{2} = E\left\{\frac{1}{n}\sum_{i=1}^{n}\eta_{i}(\boldsymbol{z})\eta_{i}(\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}')\right\}^{2}$$

$$= \frac{1}{n}\left\{G_{\eta}(\boldsymbol{z},\boldsymbol{z})G_{\eta}(\boldsymbol{z}',\boldsymbol{z}') + G_{\eta}^{2}(\boldsymbol{z},\boldsymbol{z}') + \sum_{k=1}^{\infty}\lambda_{k}^{2}E(\xi_{1k}^{4} - 3)\psi_{k}^{2}(\boldsymbol{z})\psi_{k}^{2}(\boldsymbol{z}')\right\}.$$

Therefore, $E\left\{\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}')\right\}^{2} \approx n^{-1}$. Hence, following from Bernstein inequality, $\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}}\left|\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}') - G_{\eta}(\boldsymbol{z},\boldsymbol{z}')\right| = O_{P}\left\{n^{-1/2}(\log n)^{1/2}\right\}$, and the desired result follows.

Proof of Theorem 4. Note that

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2}|\widehat{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}')-G_{\eta}(\boldsymbol{z},\boldsymbol{z}')|\leq \sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2}\{|\widehat{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}')-\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}')|+|\widetilde{G}_{\eta}(\boldsymbol{z},\boldsymbol{z}')-G_{\eta}(\boldsymbol{z},\boldsymbol{z}')|\},$$

where $\sup_{(z,z')\in\Omega^2}|\widetilde{G}_{\eta}(z,z')-G_{\eta}(z,z')|=o_P(1)$ according to Lemma A.18, and

$$\begin{split} \sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} & |\widehat{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}') - \widetilde{G}_{\eta}(\boldsymbol{z}, \boldsymbol{z}')| \leq \sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n \eta_i(\boldsymbol{z}) D_i(\boldsymbol{z}') \right| \\ & + \sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n \eta_i(\boldsymbol{z}') D_i(\boldsymbol{z}) \right| + \sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n D_i(\boldsymbol{z}) D_i(\boldsymbol{z}') \right|. \end{split}$$

With some simple calculations, we have

$$\sum_{i=1}^n \eta_i(\boldsymbol{z}) D_i(\boldsymbol{z}') = \sum_{i=1}^n \eta_i(\boldsymbol{z}) \widetilde{b}_i(\boldsymbol{z}') + \sum_{i=1}^n \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') + \sum_{i=1}^n \eta_i(\boldsymbol{z}) \widetilde{\varepsilon}_i(\boldsymbol{z}'),$$

where $\nabla \eta_i = \widetilde{\eta}_i - \eta_i$. According to (A.39), (A.42) and (A.47), we have

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n \eta_i(\boldsymbol{z}) D_i(\boldsymbol{z}') + n^{-1} \sum_{i=1}^n \eta_i(\boldsymbol{z}') D_i(\boldsymbol{z}) \right| = o_P(1).$$

Note that

$$\sum_{i=1}^{n} D_{i}(\boldsymbol{z}) D_{i}(\boldsymbol{z}') = \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \widetilde{b}_{i}(\boldsymbol{z}') + \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') + \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') + \sum_{i=1}^{n} \widetilde{\varepsilon}_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') + \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') + \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}').$$

It follows from (A.38), (A.41), (A.43)–(A.46) that $\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2}|n^{-1}\sum_{i=1}^nD_i(\boldsymbol{z})D_i(\boldsymbol{z}')|=o_P(1)$. The desired result is established.

Lemma A.19. Under Assumptions (A1)–(A5), (C1)–(C3), we have

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n \widetilde{b}_i(\boldsymbol{z}) \widetilde{b}_i(\boldsymbol{z}') \right| = O_P\{n^{-1} |\Delta_{\eta}|^{-2} (\log n)^{1/2}\}, \tag{A.38}$$

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| \sum_{i=1}^n \eta_i(\boldsymbol{z}) \widetilde{b}_i(\boldsymbol{z}') \right| = O_P\{n^{-1} (\log n)^{1/2}\}. \tag{A.39}$$

Proof. According to (A.19) and (A.37), we have

$$\widetilde{b}_{i}(\boldsymbol{z}) = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sum_{\ell=0}^{p} X_{i\ell} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\ell}(\boldsymbol{z}_{j}) \}$$

$$= \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sum_{\ell=0}^{p} X_{i\ell} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\mu,\ell}(\boldsymbol{z}_{j}) - \widehat{\eta}_{\ell}(\boldsymbol{z}_{j}) - \widehat{\varepsilon}_{\ell}(\boldsymbol{z}_{j}) \}. \quad (A.40)$$

Thus,

$$\frac{1}{n} \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \widetilde{b}_{i}(\boldsymbol{z}') = \frac{1}{n} \sum_{i=1}^{n} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \left[\frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \sum_{\ell=0}^{p} X_{i\ell} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\ell}(\boldsymbol{z}_{j}) \} \right] \\
\times \left[\frac{1}{N} \sum_{j'=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \sum_{\ell'=0}^{p} X_{i\ell'} \{ \beta_{\ell'}^{o}(\boldsymbol{z}_{j'}) - \widehat{\beta}_{\ell'}(\boldsymbol{z}_{j'}) \} \right] \boldsymbol{\Upsilon}_{n}^{-1} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}') \\
\approx \frac{1}{n |\Delta_{\eta}|^{4}} \sum_{i=1}^{n} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top} \left[\frac{1}{N^{2}} \sum_{j,j'=1}^{N} \widetilde{\mathbf{B}}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top} \sum_{\ell,\ell'=0}^{p} X_{i\ell} X_{i\ell'} \{ \beta_{\ell}^{o}(\boldsymbol{z}_{j}) - \widehat{\beta}_{\ell}(\boldsymbol{z}_{j}) \} \\
\times \{ \beta_{\ell'}^{o}(\boldsymbol{z}_{j'}) - \widehat{\beta}_{\ell'}(\boldsymbol{z}_{j'}) \} \right] \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}').$$

Therefore, by Theorem 1, we have

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\widetilde{b}_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\right\} \approx \sum_{\ell=0}^{p}\sum_{\ell'=0}^{p}|\Delta_{\eta}|^{-2}\|\beta_{\ell}^{o}-\widehat{\beta}_{\ell}\|\|\beta_{\ell'}^{o}-\widehat{\beta}_{\ell'}\| \approx n^{-1}|\Delta_{\eta}|^{-2}.$$

We have
$$E\left\{n^{-1}\sum_{i=1}^{n}\widetilde{b}_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\right\}^{2} = \frac{1}{n^{2}}\sum_{i,i'=1}^{n}E\left\{\widetilde{b}_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z})\widetilde{b}_{i'}(\boldsymbol{z}')\right\}$$
, where
$$E\left\{\widetilde{b}_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z})\widetilde{b}_{i'}(\boldsymbol{z}')\right\} \simeq |\Delta_{\eta}|^{-8}$$

$$\times E\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top}\left[\frac{1}{N^{2}}\sum_{j,j'=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top}\sum_{\ell,\ell'=0}^{p}X_{i\ell}X_{i\ell'}\{\beta_{\ell}^{o}(\boldsymbol{z}_{j})-\widehat{\beta}_{\ell}(\boldsymbol{z}_{j})\}\{\beta_{\ell'}^{o}(\boldsymbol{z}_{j'})-\widehat{\beta}_{\ell'}(\boldsymbol{z}_{j'})\}\right]\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')$$

$$\times \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top}\left[\frac{1}{N^{2}}\sum_{j,j'=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top}\sum_{\ell,\ell'=0}^{p}X_{i'\ell}X_{i'\ell'}\{\beta_{\ell}^{o}(\boldsymbol{z}_{j})-\widehat{\beta}_{\ell}(\boldsymbol{z}_{j})\}\{\beta_{\ell'}^{o}(\boldsymbol{z}_{j'})-\widehat{\beta}_{\ell'}(\boldsymbol{z}_{j'})\}\right]\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')$$

$$\approx n^{-2}|\Delta_{\eta}|^{-4}.$$

Thus, (A.38) follows from the Bernstein inequality after the discretization. Following from (A.40), we have, for any $i, i' = 1, \ldots, n$,

$$\begin{split} &E\left\{\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z}')\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\right\} \\ &\simeq |\triangle_{\eta}|^{-4}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top}\frac{1}{N^{2}}\sum_{j,j'=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top}E\left\{\sum_{\ell,\ell'=0}^{p}X_{i\ell}X_{i\ell'}\widehat{\eta}_{\ell}(\boldsymbol{z}_{j})\widehat{\eta}_{\ell'}(\boldsymbol{z}_{j'})\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\right\}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}'),\\ &E\left\{\sum_{\ell,\ell'=0}^{p}X_{i\ell}X_{i\ell'}\widehat{\eta}_{\ell}(\boldsymbol{z}_{j})\widehat{\eta}_{\ell'}(\boldsymbol{z}_{j'})\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\right\} = \frac{1}{n^{2}N^{2}}\sum_{i'',i'''=1}^{n}E\left[\left\{\mathbf{X}_{i}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right\}^{\top}\Gamma_{n,\rho}^{-1}\\ &\times\sum_{j'',j'''=1}^{N}\mathbf{X}_{i''}\mathbf{X}_{i'''}^{\top}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j''})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'''})^{\top}\Gamma_{n,\rho}^{-1}\mathbf{X}_{i'}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right]E\left\{\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\eta_{i''}(\boldsymbol{z}_{j''})\eta_{i'''}(\boldsymbol{z}_{j'''})\right\}\\ &=\frac{1}{n^{2}N^{2}}E\left[\left\{\mathbf{X}_{i}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\right\}^{\top}\Gamma_{n,\rho}^{-1}\sum_{j'',j'''=1}^{N}\mathbf{X}_{i''}\mathbf{X}_{i'''}^{\top}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j''})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'''})\widetilde{\mathbf{T}}\Gamma_{n,\rho}^{-1}\left\{\mathbf{X}_{i'}\otimes\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right\}\right]\\ &\times E\left\{\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\eta_{i}(\boldsymbol{z}_{j''})\eta_{i'}(\boldsymbol{z}_{j'''})+\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\eta_{i}(\boldsymbol{z}_{j'''})\eta_{i}(\boldsymbol{z}_{j'''})\right\}\simeq n^{-2}. \end{split}$$

$$\text{Therefore, } E\left\{\frac{1}{n}\sum_{i=1}^{n}\eta_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\right\}^{2} &= \frac{1}{n^{2}}\sum_{i,i'=1}^{n}E\left\{\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z}')\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\right\} = O(n^{-2}). \end{split}$$

Therefore,
$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\eta_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\right\}^{2} = \frac{1}{n^{2}}\sum_{i,i'=1}^{n}E\{\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z}')\eta_{i}(\boldsymbol{z})\eta_{i'}(\boldsymbol{z})\} = O(n^{-2}).$$

Lemma A.20. Under Assumptions (A1)–(A5), (C1)–(C3), we have

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| n^{-1} \sum_{i=1}^n \nabla \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') \right| = O_P \left\{ |\triangle_{\eta}|^{2(s+1)} \sum_{k=1}^{K_n} \lambda_k \|\psi_k\|_{s+1, \infty}^2 + \sum_{k=K_n+1}^{\infty} \lambda_k \|\psi_k\|_{\infty}^2 \right\}, \tag{A.41}$$

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2} \left| n^{-1} \sum_{i=1}^n \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') \right| = O_P \left\{ |\Delta_{\eta}|^{s+1} \sum_{k=1}^{K_n} \lambda_k \|\psi_k\|_{s+1,\infty} \|\psi_k\|_{\infty} + \sum_{k=K_n+1}^{\infty} \lambda_k \|\psi_k\|_{\infty}^2 \right\},$$
(A.42)

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^{2}} \left| \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \widetilde{b}_{i}(\boldsymbol{z}') \right| = O_{P} \left\{ (\log n)^{1/2} n^{-1} |\Delta_{\eta}|^{s+1} \sum_{k=1}^{K_{n}} \lambda_{k} \|\psi_{k}\|_{s+1, \infty} \|\psi_{k}\|_{\infty} \right\} + O_{P} \left\{ (\log n)^{1/2} n^{-1} \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2} \right\}. \tag{A.43}$$

Proof. For any $k \geq 1$, denote $\widetilde{\psi}_k(\boldsymbol{z}) = \widetilde{\mathbf{B}}(\boldsymbol{z})^{\top} \boldsymbol{\Upsilon}_n^{-1} \frac{1}{N} \sum_{j=1}^N \widetilde{\mathbf{B}}(\boldsymbol{z}_j) \psi_k(\boldsymbol{z}_j)$, and $\nabla \psi_k = \widetilde{\psi}_k - \psi_k$. According to Assumption (C2), we hav(C3)e, for any $k \geq 1$, $\|\nabla \psi_k\|_{\infty} \leq C|\Delta_{\eta}|^{s+1} \|\psi_k\|_{s+1,\infty}$ and $\|\widetilde{\psi}_k\|_{\infty} \leq \|\psi_k\|_{\infty} + \|\nabla \psi_k\|_{\infty} \leq 2\|\psi_k\|_{\infty}$, as $n \to \infty$. It is easy to see that $\nabla \eta_i(\boldsymbol{z}') = \sum_{k=1}^{\infty} \lambda_k^{1/2} \xi_{ik} \nabla \psi_k(\boldsymbol{z}')$.

We first show (A.41). Let $\bar{\xi}_{\cdot kk'} = n^{-1} \sum_{i=1}^{n} \xi_{ik} \xi_{ik'}$, where $E(\bar{\xi}_{\cdot kk'}) = I(k = k')$ and $E(\bar{\xi}_{\cdot kk'})^2 \leq (E\xi_{ik}^4 E\xi_{ik'}^4)^{1/2} \leq C$. Simple calculation yields that $\frac{1}{n} \sum_{i=1}^{n} \nabla \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') = \sum_{k,k'=1}^{\infty} \bar{\xi}_{\cdot kk'} (\lambda_k \lambda_{k'})^{1/2} \nabla \psi_k(\boldsymbol{z}) \nabla \psi_{k'}(\boldsymbol{z}')$. Thus, by Assumption (C2), we have

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| E\left\{ \frac{1}{n} \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') \right\} \right| = \sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| \sum_{k=1}^{\infty} \lambda_{k} \nabla \psi_{k}(\boldsymbol{z}) \nabla \psi_{k}(\boldsymbol{z}') \right|$$

$$\leq |\triangle_{\eta}|^{2(s+1)} \sum_{k=1}^{K_{n}} \lambda_{k} \|\psi_{k}\|_{s+1,\infty}^{2} + C_{\psi} \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2}.$$

In addition, we have

$$\sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} E\left\{\nabla \eta_{i}(\boldsymbol{z})\nabla \eta_{i}(\boldsymbol{z}')\right\}^{2} = \sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} E\left[\sum_{k=1}^{\infty} \xi_{ik}^{2} \lambda_{k} (\nabla \psi_{k})^{2}(\boldsymbol{z}) \sum_{k'=1}^{\infty} \xi_{ik'}^{2} \lambda_{k'} (\nabla \psi_{k'})^{2}(\boldsymbol{z}')\right]$$
$$\approx n^{-1} \left\{|\Delta_{\eta}|^{2(s+1)} \sum_{k=1}^{K_{n}} \lambda_{k} \|\psi_{k}\|_{s+1,\infty}^{2} + \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2}\right\}^{2}.$$

Thus,

$$\sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} \operatorname{Var}\left\{\frac{1}{n}\sum_{i=1}^n \nabla \eta_i(\boldsymbol{z})\nabla \eta_i(\boldsymbol{z}')\right\} \asymp \left\{|\triangle_{\eta}|^{2(s+1)}\sum_{k=1}^{K_n} \lambda_k \|\psi_k\|_{s+1,\infty}^2 + \sum_{k=K_n+1}^{\infty} \lambda_k \|\psi_k\|_{\infty}^2\right\}^2.$$

Therefore, using the discretization method and Bernstein inequality

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| \frac{1}{n} \sum_{i=1}^n \nabla \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') - E\{\nabla \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}')\} \right|$$

$$= O_P \left\{ (\log n)^{1/2} n^{-1/2} |\triangle_{\eta}|^{2(s+1)} \sum_{k=1}^{K_n} \lambda_k ||\psi_k||_{s+1,\infty}^2 (\log n)^{1/2} n^{-1/2} \sum_{k=K_n+1}^{\infty} \lambda_k ||\psi_k||_{\infty}^2 \right\}.$$

Next we derive (A.42). Noting that $n^{-1} \sum_{i=1}^{n} \eta_i(\boldsymbol{z}) \nabla \eta_i(\boldsymbol{z}') = \xi_{\cdot kk'}(\lambda_k \lambda_{k'})^{1/2} \psi_k(\boldsymbol{z}') (\nabla \psi_{k'})(\boldsymbol{z}')$, we have

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| E\left\{ \frac{1}{n} \sum_{i=1}^{n} \eta_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') \right\} \right| \leq \sum_{k=1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty} \|\nabla \psi_{k'}\|_{\infty}$$

$$\leq C |\Delta_{\eta}|^{s+1} \sum_{k=1}^{K_{n}} \lambda_{k} \|\psi_{k}\|_{s+1,\infty} \|\psi_{k}\|_{\infty} + \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2},$$

$$\operatorname{var} \left\{ n^{-1} \sum_{i=1}^{n} \eta_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') \right\} = n^{-1} \left[E\left\{ \eta_{i}^{2}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}')^{2} \right\} - \left\{ E \eta_{i}(\boldsymbol{z}) \nabla \eta_{i}(\boldsymbol{z}') \right\}^{2} \right],$$

$$\sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} E\left\{\eta_i^2(\boldsymbol{z})\nabla\eta_i(\boldsymbol{z}')^2\right\} = \sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} \left\{ \sum_{k=1}^{\infty} E\xi_{ik}^4 \lambda_k^2 \psi_k^2(\boldsymbol{z})(\nabla\psi_k)^2(\boldsymbol{z}') + \sum_{k\neq k'} \lambda_k \lambda_{k'} \psi_k^2(\boldsymbol{z})(\nabla\psi_k)^2(\boldsymbol{z}') \right\} \\
\leq C\left\{ |\Delta_{\eta}|^{2(s+1)} \sum_{k=1}^{K_n} \lambda_k \|\psi_k\|_{s+1,\infty}^2 + \sum_{k=K_n+1}^{\infty} \lambda_k \|\psi_k\|_{\infty}^2 \right\},$$

and

$$\sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} |E\left\{\eta_i(\boldsymbol{z})\nabla\eta_i(\boldsymbol{z}')\right\}| \leq C\left\{|\triangle_{\eta}|^{s+1}\sum_{k=1}^{K_n}\lambda_k\|\psi_k\|_{s+1,\infty}\|\psi_k\|_{\infty} + \sum_{k=K_n+1}^{\infty}\lambda_k\|\psi_k\|_{\infty}^2\right\}.$$

Therefore,

$$\sup_{\boldsymbol{z},\boldsymbol{z}'\in\Omega} E\left\{n^{-1}\sum_{i=1}^{n} \eta_{i}(\boldsymbol{z})\nabla\eta_{i}(\boldsymbol{z}')\right\}^{2} = O\left[\left\{|\triangle_{\eta}|^{s+1}\sum_{k=1}^{K_{n}} \lambda_{k}\|\psi_{k}\|_{s+1,\infty}\|\psi_{k}\|_{\infty} + \sum_{k=K_{n}+1}^{\infty} \lambda_{k}\|\psi_{k}\|_{\infty}^{2}\right\}^{2}\right].$$

Hence,

$$\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^{2}} \left| n^{-1} \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') - E \left\{ \nabla \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right\} \right|$$

$$= O_{P} \left\{ (\log n)^{1/2} n^{-1/2} |\Delta_{\eta}|^{2(s+1)} \sum_{k=1}^{K_{n}} \lambda_{k} \|\psi_{k}\|_{s+1,\infty}^{2} + (\log n)^{1/2} n^{-1/2} \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2} \right\}$$

using the discretization method and Bernstein inequality.

Finally, we provide the proof of (A.43). Note that

$$E\left\{\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z}')\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i'}(\boldsymbol{z})\right\} \simeq |\triangle_{\eta}|^{-4}$$

$$\times \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top} \frac{1}{N^{2}} \sum_{j,j'=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'})^{\top} E\left\{\sum_{\ell,\ell'=0}^{p} X_{i\ell} X_{i\ell'} \widehat{\eta}_{\ell}(\boldsymbol{z}_{j}) \widehat{\eta}_{\ell'}(\boldsymbol{z}_{j'})\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i'}(\boldsymbol{z})\right\} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}'),$$

and by (A.42),

$$E\left\{\sum_{\ell,\ell'=0}^{p} X_{i\ell} X_{i\ell'} \widehat{\eta}_{\ell}(\boldsymbol{z}_{j}) \widehat{\eta}_{\ell'}(\boldsymbol{z}_{j'}) \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i'}(\boldsymbol{z})\right\} = \frac{1}{n^{2}N^{2}} \sum_{i'',i'''=1}^{n} E\left[\left\{\mathbf{X}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\right\}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \times \sum_{j'',j'''=1}^{N} \mathbf{X}_{i''} \mathbf{X}_{i'''}^{\top} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j''}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'''})^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \mathbf{X}_{i'} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right] E\left\{\eta_{i''}(\boldsymbol{z}_{j''}) \eta_{i'''}(\boldsymbol{z}_{j'''}) \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i'}(\boldsymbol{z})\right\}$$

$$= \frac{1}{n^{2}N^{2}} E\left[\left\{\mathbf{X}_{i} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\right\}^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \sum_{j'',j'''=1}^{N} \mathbf{X}_{i''} \mathbf{X}_{i'''}^{\top} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j''}) \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'''})^{\top} \boldsymbol{\Gamma}_{n,\rho}^{-1} \left\{\mathbf{X}_{i'} \otimes \widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})\right\}\right]$$

$$\times E\left\{\eta_{i}(\boldsymbol{z}_{j''}) \eta_{i'}(\boldsymbol{z}_{j'''}) \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i'}(\boldsymbol{z}) + \eta_{i'}(\boldsymbol{z}_{j''}) \eta_{i}(\boldsymbol{z}_{j'''}) \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i'}(\boldsymbol{z})\right\}.$$

If $i \neq i'$, we have

$$E\left\{\eta_{i}(\boldsymbol{z}_{j''})\eta_{i'}(\boldsymbol{z}_{j'''})\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i'}(\boldsymbol{z}) + \eta_{i'}(\boldsymbol{z}_{j''})\eta_{i}(\boldsymbol{z}_{j'''})\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i'}(\boldsymbol{z})\right\}$$

$$\approx \left\{\sum_{k=1}^{K_{n}}\lambda_{k}|\triangle|_{\eta}^{s+1}\|\psi_{k}\|_{s+1,\infty}\|\psi_{k}\|_{\infty} + \sum_{k=K_{n}+1}^{\infty}\lambda_{k}\|\psi_{k}\|_{\infty}^{2}\right\}^{2}.$$

If i = i', then we have

$$E\left\{\eta_{i}(\boldsymbol{z}_{j''})\eta_{i}(\boldsymbol{z}_{j'''})\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i}(\boldsymbol{z})\right\} = \sum_{k=1}^{\infty} \lambda_{k}^{2} E\xi_{ik}^{4} \psi_{k}(\boldsymbol{z}'')\psi_{k}(\boldsymbol{z}''')\nabla\psi_{k}(\boldsymbol{z})\nabla\psi_{k}(\boldsymbol{z})$$

$$\leq \sum_{k=1}^{K_{n}} \lambda_{k}^{2} |\Delta|_{\eta}^{2s+2} ||\psi_{k}||_{s+1,\infty}^{2} ||\psi_{k}||_{\infty}^{2} + \sum_{k=K_{n}+1}^{\infty} \lambda_{k}^{2} ||\psi_{k}||_{\infty}^{4}.$$

Thus,

$$E\left\{\sum_{\ell,\ell'=0}^{p} X_{i\ell} X_{i\ell'} \widehat{\eta}_{\ell}(\boldsymbol{z}_{j}) \widehat{\eta}_{\ell'}(\boldsymbol{z}_{j'}) \nabla \eta_{i}(\boldsymbol{z}) \nabla \eta_{i'}(\boldsymbol{z})\right\}$$

$$\approx \left\{\sum_{k=1}^{K_{n}} \lambda_{k} |\Delta|_{\eta}^{s+1} \|\psi_{k}\|_{s+1,\infty} \|\psi_{k}\|_{\infty} + \sum_{k=K_{n}+1}^{\infty} \lambda_{k} \|\psi_{k}\|_{\infty}^{2}\right\}^{2}.$$

Therefore,

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\nabla\eta_{i}(\boldsymbol{z})\widetilde{b}_{i}(\boldsymbol{z}')\right\}^{2} = \frac{1}{n^{2}}\sum_{i,i'=1}^{n}E\widetilde{b}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z}')\nabla\eta_{i}(\boldsymbol{z})\nabla\eta_{i'}(\boldsymbol{z})$$

$$= O\left[n^{-2}\sum_{k=1}^{K_{n}}\lambda_{k}^{2}|\Delta|_{\eta}^{2s+2}\|\psi_{k}\|_{s+1,\infty}^{2}\|\psi_{k}\|_{\infty}^{2} + n^{-2}\sum_{k=K_{n}+1}^{\infty}\lambda_{k}^{2}\|\psi_{k}\|_{\infty}^{4}\right].$$

Thus, (A.43) is obtained.

Lemma A.21. Under Assumptions (A1)–(A5), (C1)–(C3), we have

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| n^{-1} \sum_{i=1}^{n} \widetilde{\varepsilon}_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right| = O_{P}(N^{-1}|\Delta_{\eta}|^{-2}), \tag{A.44}$$

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| n^{-1} \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right| = O_{P} \left\{ n^{-1/2} N^{-1/2} (\log n)^{1/2} |\Delta_{\eta}|^{s} \sum_{k=1}^{K_{n}} \lambda_{k}^{1/2} \|\psi_{k}\|_{s+1,\infty} \right\}$$

$$+ O_{P} \left\{ n^{-1/2} N^{-1/2} |\Delta_{\eta}|^{-1} (\log n)^{1/2} \sum_{k=K_{n}+1}^{\infty} \lambda_{k}^{1/2} \|\psi_{k}\|_{\infty} \right\}, \tag{A.45}$$

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| n^{-1} \sum_{i=1}^{n} \widetilde{b}_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right| = O_{P} \left\{ n^{-1} N^{-1} |\Delta|^{-2} (\log n)^{1/2} \right\}, \tag{A.46}$$

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^{2}} \left| n^{-1} \sum_{i=1}^{n} \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right| = O_{P} \left\{ n^{-1/2} N^{-1/2} |\Delta_{\eta}|^{-1} (\log n)^{1/2} \right\}. \tag{A.47}$$

Proof. We first show (A.44). Let $\bar{\varepsilon}_{.jj'} = n^{-1} \sum_{i=1}^{n} \varepsilon_{ij} \varepsilon_{ij'}$, where $E(\bar{\varepsilon}_{.jj'}) = I(j = j')$. Note that

(A.47)

$$\frac{1}{n}\sum_{i=1}^{n}\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}') = \widetilde{\mathbf{B}}(\boldsymbol{z})^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\left\{\frac{1}{N^{2}}\sum_{j=1}^{N}\sum_{j'=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top}\boldsymbol{\sigma}(\boldsymbol{z}_{j})\boldsymbol{\sigma}(\boldsymbol{z}_{j'})\bar{\varepsilon}_{\cdot jj'}\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}(\boldsymbol{z}').$$

It is easy to see that,

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\} = \widetilde{\mathbf{B}}(\boldsymbol{z})^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\left\{\frac{1}{N^{2}}\sum_{j=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j'})^{\top}\sigma^{2}(\boldsymbol{z}_{j})\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}(\boldsymbol{z}').$$

Therefore, $\sup_{(\boldsymbol{z}, \boldsymbol{z}') \in \Omega^2} \left| E\left\{ \frac{1}{n} \sum_{i=1}^n \widetilde{\varepsilon}_i(\boldsymbol{z}) \widetilde{\varepsilon}_i(\boldsymbol{z}') \right\} \right| = O(N^{-1} |\triangle_{\eta}|^{-2})$. In addition, note that

$$E\left\{\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\} = \widetilde{\mathbf{B}}(\boldsymbol{z})^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\left\{\frac{1}{N^{2}}\sum_{j=1}^{N}\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}(\boldsymbol{z}_{j})^{\top}\sigma^{2}(\boldsymbol{z}_{j})\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}(\boldsymbol{z}') = O(N^{-1}|\Delta_{\eta}|^{-2}),$$

$$E\left\{\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\}^{2} = E\left[\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z})^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\left\{\frac{1}{N^{2}}\sum_{j=1}^{N}\sum_{j'=1}^{N}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'})^{\top}\sigma(\boldsymbol{z}_{j})\sigma(\boldsymbol{z}_{j'})\varepsilon_{ij}\varepsilon_{ij'}\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')\right]^{2}$$

$$\approx \frac{|\Delta_{\eta}|^{-8}}{N^{4}}\sum_{j,j',j'',j'''=1}^{N}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'})^{\top}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j''})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j''})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'''})^{\top}$$

$$\times \sigma(\boldsymbol{z}_{j})\sigma(\boldsymbol{z}_{j'})\sigma(\boldsymbol{z}_{j''})\sigma(\boldsymbol{z}_{j'''})\varepsilon_{ij}\varepsilon_{ij'}\varepsilon_{ij''}\varepsilon_{ij'''}\varepsilon_{ij'''} \approx N^{-2}|\Delta_{\eta}|^{-4}.$$

Thus, $\operatorname{var}\left\{\frac{1}{n}\sum_{i=1}^{n}\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\} = \frac{1}{n^{2}}\sum_{i=1}^{n}\operatorname{var}\left\{\widetilde{\varepsilon}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\} \asymp n^{-1}N^{-2}|\Delta_{\eta}|^{-4}$. Therefore,

$$\sup_{(\boldsymbol{z},\boldsymbol{z}')\in\Omega^2} \left| n^{-1} \sum_{i=1}^n \widetilde{\varepsilon}_i(\boldsymbol{z}) \widetilde{\varepsilon}_i(\boldsymbol{z}') - E\left\{ \widetilde{\varepsilon}_i(\boldsymbol{z}) \widetilde{\varepsilon}_i(\boldsymbol{z}') \right\} \right| = O_P\left\{ n^{-1/2} N^{-1} (\log n)^{1/2} |\Delta_{\eta}|^{-2} \right\}$$

using the discretization method and Bernstein inequality.

Next we derive (A.45). Note that

$$\frac{1}{n} \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \xi_{ik} \lambda_{k}^{1/2} \nabla \psi_{k}(\boldsymbol{z}) \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \left\{ \frac{1}{N} \sum_{j=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j}) \sigma(\boldsymbol{z}_{j}) \varepsilon_{ij} \right\},$$

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \nabla \eta_{i}(\boldsymbol{z}) \widetilde{\varepsilon}_{i}(\boldsymbol{z}') \right\}^{2} = \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \xi_{ik} \xi_{i'k'} (\lambda_{k} \lambda_{k'})^{1/2} \nabla \psi_{k}(\boldsymbol{z})$$

$$\times \nabla \psi_{k'}(\boldsymbol{z}) \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top} \boldsymbol{\Upsilon}_{n}^{-1} \left\{ \frac{1}{N^{2}} \sum_{j=1}^{N} \sum_{j'=1}^{N} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j}) \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'})^{\top} \sigma(\boldsymbol{z}_{j}) \sigma(\boldsymbol{z}_{j'}) \varepsilon_{ij} \varepsilon_{i'j'} \right\} \boldsymbol{\Upsilon}_{n}^{-1} \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}').$$

Next observe that $E\left[\frac{1}{n}\sum_{i=1}^{n}\nabla\eta_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right]=0$ and

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\nabla\eta_{i}(\boldsymbol{z})(\nabla\psi_{k})^{2}(\boldsymbol{z})\right\} = \frac{1}{n^{2}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\lambda_{k}(\nabla\psi_{k})^{2}(\boldsymbol{z})$$
$$\times\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\left\{\frac{1}{N^{2}}\sum_{j=1}^{N}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})^{\top}\boldsymbol{\sigma}^{2}(\boldsymbol{z}_{j})\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')$$

So,

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\nabla\eta_{i}(z)(\nabla\psi_{k})^{2}(z)\right\} \leq \frac{C_{1}|\triangle_{\eta}|^{-2}}{nN}\left\{|\triangle_{\eta}|^{2(s+1)}\sum_{k=1}^{K_{n}}\lambda_{k}\|\psi_{k}\|_{s+1,\infty}^{2} + \sum_{k=K_{n}+1}^{\infty}\lambda_{k}\|\psi_{k}\|_{\infty}^{2}\right\}.$$

Thirdly, we prove (A.46). Note that for any $i,\,i',\,j,\,j',$ we have

$$E\left\{\widetilde{b}_{i}(\boldsymbol{z})\varepsilon_{ij}\widetilde{b}_{i'}(\boldsymbol{z})\varepsilon_{i'j'}\right\}$$

$$=E\left[\mathbf{B}_{\eta}(\boldsymbol{z})^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\frac{1}{N^{2}}\sum_{j'',j'''=1}^{N}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j''})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j'''})^{\top}\sum_{\ell,\ell'=0}^{p}X_{i\ell}\widehat{\varepsilon}_{\ell}(\boldsymbol{z}_{j''})X_{i\ell'}\widehat{\varepsilon}_{\ell'}(\boldsymbol{z}_{j'''})\varepsilon_{ij}\varepsilon_{i'j'}\boldsymbol{\Upsilon}_{n}^{-1}\mathbf{B}_{\eta}(\boldsymbol{z})\right]$$

$$=O(n^{-2}N^{-2}|\Delta|^{-4}).$$

Therefore,

$$E\left\{\widetilde{b}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z})\widetilde{\varepsilon}_{i'}(\boldsymbol{z}')\right\} = \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top}\boldsymbol{\Upsilon}_{n}^{-1}\frac{1}{N^{2}}\sum_{j,j'=1}^{N}E\left\{\widetilde{b}_{i}(\boldsymbol{z})\varepsilon_{ij}\widetilde{b}_{i'}(\boldsymbol{z})\varepsilon_{i'j'}\right\}\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')$$

$$= O(n^{-2}N^{-2}|\Delta|^{-4}),$$

$$E\left[n^{-1}\sum_{i=1}^{n}\widetilde{b}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right]^{2} = \frac{1}{n^{2}}\sum_{i,i'=1}^{n}E\left\{\widetilde{b}_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\widetilde{b}_{i'}(\boldsymbol{z})\widetilde{\varepsilon}_{i'}(\boldsymbol{z}')\right\} = O(n^{-2}N^{-2}|\Delta|^{-4}).$$

Finally, we show (A.47). Note that

$$\sum_{i=1}^n \eta_i(\boldsymbol{z}) \widetilde{\varepsilon}_i(\boldsymbol{z}') = \sum_{i=1}^n \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top} \boldsymbol{\Upsilon}_n^{-1} \frac{1}{N} \sum_{j=1}^N \widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_j) \sigma(\boldsymbol{z}_j) \varepsilon_{ij} \sum_{k=1}^{\infty} \xi_{ik} \lambda_k^{1/2} \psi_k(\boldsymbol{z}),$$

where $E\left\{n^{-1}\sum_{i=1}^{n}\eta_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\}=0$, and

$$E\left\{n^{-1}\sum_{i=1}^{n}\eta_{i}(\boldsymbol{z})\widetilde{\varepsilon}_{i}(\boldsymbol{z}')\right\}^{2} = n^{-1}E\{\eta_{i}(\boldsymbol{z})^{2}\}E\{\widetilde{\varepsilon}_{i}(\boldsymbol{z}')^{2}\} = n^{-1}G_{\eta}(\boldsymbol{z},\boldsymbol{z}')\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}')^{\top}$$
$$\times \boldsymbol{\Upsilon}_{n}^{-1}\frac{1}{N^{2}}\sum_{j=1}^{N}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}_{j})^{\top}\sigma^{2}(\boldsymbol{z}_{j})\boldsymbol{\Upsilon}_{n}^{-1}\widetilde{\mathbf{B}}_{\eta}(\boldsymbol{z}') = O(n^{-1}N^{-1}|\Delta_{\eta}|^{-2}).$$

Thus, (A.47) is obtained.

Appendix B

In this section, we provide some additional results from simulation studies and real application analysis.

B.1. More results of simulation studies

In Section 5.1 of the main paper, we illustrated the advantage of the proposed method over the complex horseshoe domain in Sangalli et al. (2013). Figure B.1 shows the two triangulations used for the horseshoe domain in this example. For implementation, the BPST method is conducted over triangulation, Δ_1 , while triangulation, Δ_2 , is used for PCST method. To visually compare different methods, we display the estimated coefficient functions for Case I (jump function) and Case II (smooth function) in Figures B.2 and B.3, respectively. The plots are obtained based on the setting: n = 50, $\lambda_1 = 0.2$, $\lambda_2 = 0.05$, $\sigma = 1.0$. Table B.1 summarizes the estimation results based on the noise level $\sigma = 1.0$.

From these figures, one sees that the BPST and PCST estimates are both very close to the true coefficient functions. When the true coefficient functions are smooth, BPST provides the best estimation, while when the true coefficient function contains jumps, PCST provides a better estimation. The performance of the Tensor method will be affected by the design of the coefficient function. Moreover, from Figure B.2 and B.3, one can see that even when the coefficient function is smooth across the boundary, the estimation accuracy is also affected by the domain of the true signal, especially the pixels which are closed to the boundary. The performance of the Kernel method is not affected by the design of the coefficient functions, instead, it heavily depends on the noise level

due to the three-stage structure. As the noise level increases, the Kernel estimates are getting more blurred.

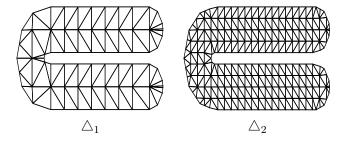


Figure B.1: Triangulations for the horseshoe domain.

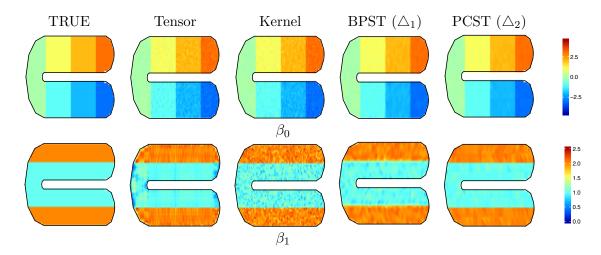


Figure B.2: True coefficient functions and their different estimators for Case I in Example 1.

In Section 5.2 of the main paper, we conduct a simulation study based on the domain of the 5th slice of the brain images illustrated in Section 6. Table B.2 demonstrates the estimation results for $\sigma=0.5$. In this example, we focus on the domain of the 35th slices of the brain image. Based on this domain, we consider two types of triangulations: Δ_5 and Δ_6 ; see Figure B.4. Table B.3 summarizes the MSE results of the BPST, kernel and tensor methods. The findings are similar to those described in Section 5.2. Tables 5.3 and B.4 summarize the ECRs of the 95% SCCs for the 5th and 35th slices, respectively, and they are all close to 95%. As the sample size increases, the ECRs are getting closer to 95%. Figures B.5 and B.6 show the true coefficient functions and an example of their estimators and 95% SCCs based on the 5th and 35th slices, respectively. The plots are generated based on the setting: n=50, $\lambda_1=0.1$, $\lambda_2=0.02$ and $\sigma=0.5$.

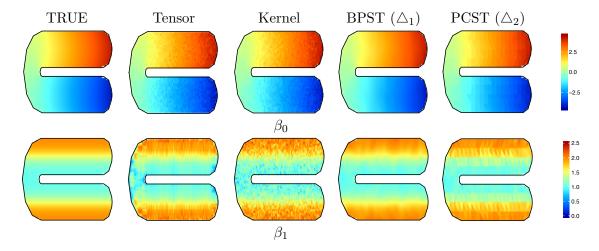


Figure B.3: True coefficient functions and their different estimators for Case II in Example 1.

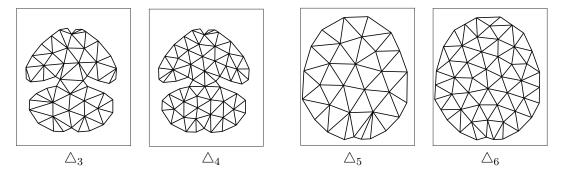


Figure B.4: Triangulations for the fifth slice $(\triangle_3, \triangle_4)$ and 35th slice $(\triangle_5, \triangle_6)$ of the brain image in Simulation Example 2.

B.2. Additional ADNI data analysis results

For the ADNI data described in Section 6 in the main paper, Table B.5 below summarizes the distribution of patients by diagnosis status and sex. Next, Figure B.7 displays the triangulations of slices used for the BPST method in the model fitting and constructing the SCCs. Finally, Figures B.8 and B.9 provide the image maps of the estimated coefficient functions for eighth, 15th, 35th, 55th, 62nd, and 65th slices, and Figures B.10 and B.11 show the corresponding significance maps. The significance maps in the eighth and 15th slice show that the increase of age increases the brain activities in the cerebellum and temporal lobe, and people with the Alzheimer's disease are more active in the cerebellum, while less active in the temporal lobe. The significance maps of the 35th slide display that the age has a negative effect on the brain activities in the

Table B.1: Estimation errors of the coefficient estimators, $\sigma = 1.0$.

Function			1 - 0.09	2 \ _ 0.006	1 - 0.2	1 - 0.05
	n	Method		$3, \lambda_2 = 0.006$		$\lambda_2 = 0.05$
Type	,,,	memod	eta_0	eta_1	β_0	β_1
		BPST	0.0059	0.0075	0.0066	0.0082
	F O	PCST	0.0023	0.0023	0.0028	0.0030
	50	Kernel	0.0201	0.0206	0.0207	0.0213
T		Tensor	0.0201	0.0132	0.0206	0.0142
Jump	100	BPST	0.0038	0.0050	0.0042	0.0054
		PCST	0.0011	0.0011	0.0014	0.0015
		Kernel	0.0100	0.0102	0.0104	0.0105
		Tensor	0.0099	0.0112	0.0103	0.0120
		BPST	0.0010	0.0012	0.0016	0.0019
	F0	PCST	0.0049	0.0065	0.0054	0.0072
	50	Kernel	0.0201	0.0206	0.0207	0.0213
C 41		Tensor	0.0189	0.0132	0.0207	0.0153
Smooth		BPST	0.0006	0.0007	0.0009	0.0010
	100	PCST	0.0037	0.0054	0.0040	0.0057
	100	Kernel	0.0100	0.0102	0.0104	0.0105
		Tensor	0.0100	0.0113	0.0103	0.0128

anterior cingulate gyrus, corpus callosum, and part of the cerebral white matter, while the female has a higher level of activities in these regions. These regions connect the left and right cerebral hemispheres and enabling communication between them. From the significance maps of the 55th, 62nd, and 65th slices, we could see an increase of brain activities in the frontal gyrus, precentral gyrus and postcentral gyrus for people with Alzheimer's disease. Our findings are consistent with the findings in the literature, see Andersen et al. (2012), Bernard and Seidler (2014), and Dubb et al. (2003).

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Table B.2: Estimation errors of the coefficient function estimators, $\sigma = 0$	Table B.2:	Estimation	errors	of the	coefficient	function	estimators.	$\sigma = 0.5$.
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n	Method	$\lambda_1 = 0$	$0.1, \ \lambda_2 =$	= 0.02	$\lambda_1 = 0.2, \ \lambda_2 = 0.05$		
		β_0	β_1	eta_2	β_0	β_3	β_2
	$BPST(\triangle_3)$	0.003	0.005	0.005	0.007	0.011	0.010
F0	$BPST(\triangle_4)$	0.003	0.005	0.005	0.006	0.009	0.009
50	Kernel	0.008	0.011	0.011	0.011	0.016	0.016
	Tensor	0.008	0.007	0.010	0.011	0.012	0.014
	$BPST(\Delta_3)$	0.002	0.002	0.002	0.003	0.005	0.005
100	$BPST(\triangle_4)$	0.002	0.002	0.002	0.003	0.004	0.004
100	Kernel	0.004	0.005	0.005	0.005	0.008	0.007
	Tensor	0.004	0.005	0.005	0.005	0.007	0.009

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Table B.3: Estima	tion errors of tr	e coemcieni	t tunction	estimators	ın tne	35th slice.

		7. /r 1 1	$\lambda_1 = 0$	$0.1, \ \lambda_2 =$	= 0.02	$\lambda_1 = 0.2, \ \lambda_2 = 0.05$			
n	σ	Method	β_0	β_1	β_2	β_0	β_1	β_2	
		$BPST(\triangle_5)$	0.003	0.005	0.005	0.007	0.011	0.011	
	0.5	$BPST(\triangle_6)$	0.003	0.005	0.005	0.007	0.011	0.010	
	0.5	Kernel	0.008	0.012	0.012	0.018	0.018	0.017	
50		Tensor	0.008	0.009	0.011	0.012	0.015	0.015	
50		$BPST(\triangle_5)$	0.003	0.005	0.005	0.007	0.011	0.011	
	1.0	$BPST(\triangle_6)$	0.003	0.005	0.005	0.007	0.011	0.011	
		Kernel	0.023	0.033	0.033	0.027	0.039	0.038	
		Tensor	0.023	0.012	0.019	0.027	0.017	0.023	
		$BPST(\triangle_5)$	0.002	0.002	0.002	0.003	0.005	0.005	
	0.5	$BPST(\triangle_6)$	0.002	0.002	0.002	0.003	0.005	0.005	
	0.5	Kernel	0.004	0.006	0.006	0.006	0.008	0.008	
100		Tensor	0.004	0.006	0.007	0.006	0.010	0.009	
100 —	$BPST(\triangle_5)$	0.002	0.002	0.002	0.003	0.005	0.005		
	1.0	$BPST(\triangle_6)$	0.002	0.002	0.002	0.003	0.005	0.005	
	1.0	Kernel	0.012	0.016	0.016	0.013	0.018	0.018	
		Tensor	0.013	0.010	0.013	0.011	0.007	0.011	

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Table B.4: The coverage rate of the 95% SCCs for the coefficient functions defined over the 35th slice.

- N			Coverage			Width		
n	Λ	σ	β_0	β_1	β_2	β_0	β_1	β_2
	(0.1.0.02)	0.5	0.962	0.916	0.934	0.307	0.344	0.347
50	(0.1, 0.02)	1.0	0.964	0.926	0.940	0.331	0.368	0.371
30	(0.2.0.05)	0.5	0.952	0.920	0.930	0.426	0.490	0.492
(0.2, 0.05)	1.0	0.96	0.920	0.934	0.449	0.512	0.512	
	(0.1.0.02)	0.5	0.956	0.952	0.940	0.214	0.240	0.244
100	(0.1, 0.02)	1.0	0.962	0.952	0.948	0.239	0.262	0.265
100 — (0	(0.2.0.05)	0.5	0.946	0.954	0.932	0.298	0.340	0.346
	(0.2,0.05)	1.0	0.952	0.954	0.938	0.317	0.359	0.365

Table B.5: Distribution of patients by diagnosis status and gender.

	CN	MCI	AD	All
Male	70	136	72	278
Female	42	77	50	169
All	112	213	122	447

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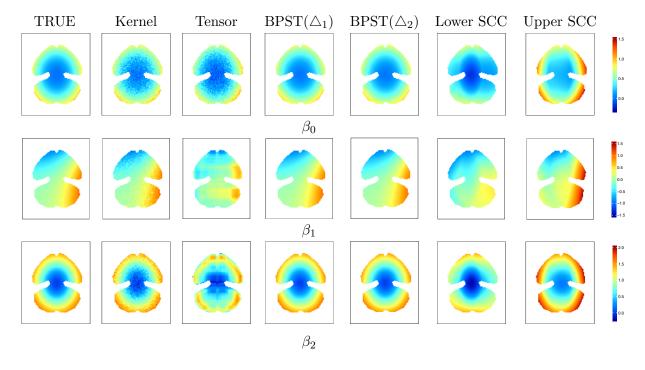


Figure B.5: True coefficient functions and their estimators and 95% SCCs based on the fifth slice.

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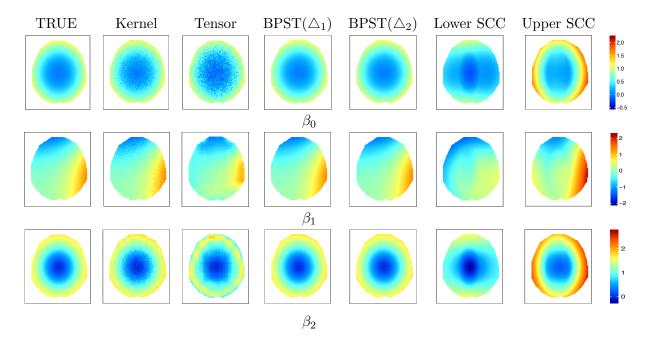


Figure B.6: True coefficient functions and their estimators and 95% SCCs based on the 35th slice.

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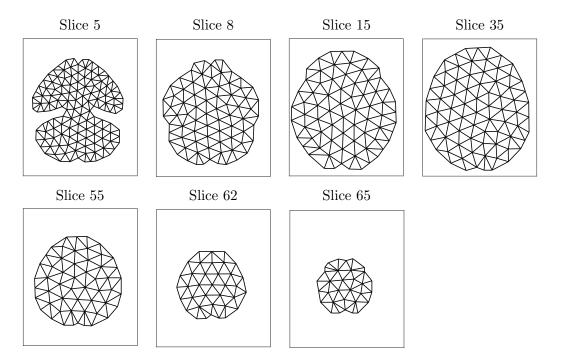


Figure B.7: Triangulation sets used in the ADNI data analysis.

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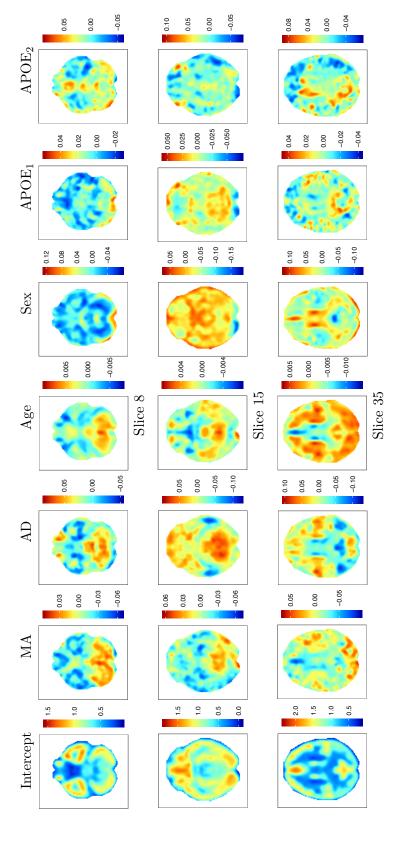


Figure B.8: The BPST estimates of the coefficient functions for the ADNI data based on the eighth, 15th and 35th slices, respectively.

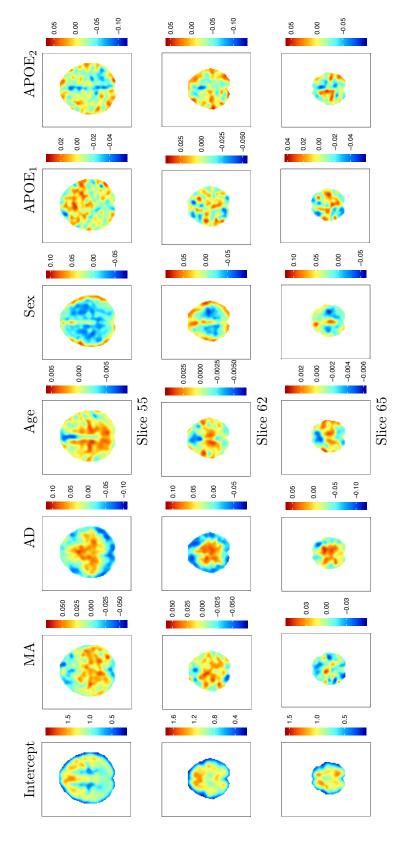


Figure B.9: The BPST estimates of the coefficient functions for the ADNI data based on the 55th, 62nd and 65th slices, respectively.

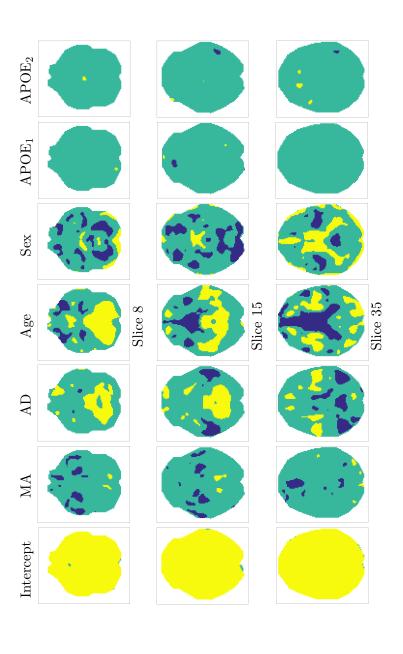


Figure B.10: The "significance" map (based on the 95% SCC) for the coefficient functions for the ADNI data. The yellow color and blue color on the map indicate the regions that zero is below the lower SCC or above the upper SCC, respectively.

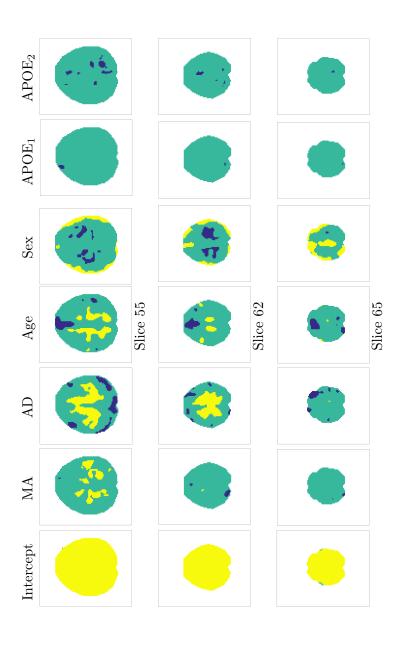


Figure B.11: The "significance" map (based on the 95% SCC) for the coefficient functions for the ADNI data. The yellow color and blue color on the map indicate the regions that zero is below the lower SCC or above the upper SCC, respectively.

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